Lecture 8: Markov random fields

Spatial Statistics and Image Analysis

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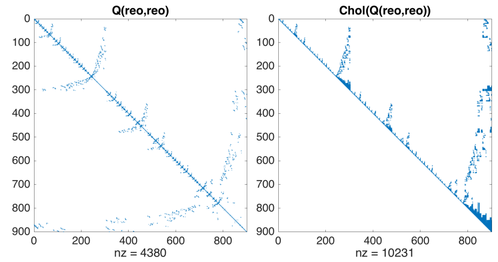
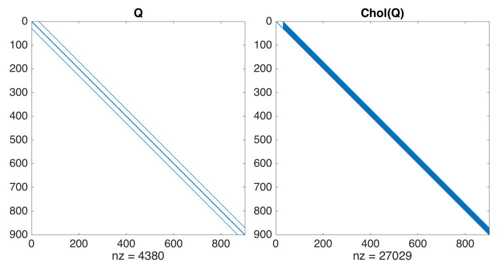
Sparsity using reorderings



- Finding the optimal reordering is an NP-hard problem, but there are many fast methods for finding good reorderings.
- The approximate minimum degree (AMD) reordering is generally a good option.
- The images above are obtained with `reo = amd(Q)` in Matlab.
- If you use reorderings, remember to also reorder the observations, covariates, etc. using the same reordering.

Sparsity of $\mathbf{Q}$ and $\mathbf{R}$



- The crucial aspect of computations with GMRFs is that the Cholesky factor $\mathbf{R}$ is sparse.
- However, it is often less sparse than the precision matrix $\mathbf{Q}$. The additional non-zero nodes is usually called fill-in.
- We can reduce the fill-in by reordering the nodes.

Computing $x = Q^{-1}v$

Three ways of computing $Q^{-1}v$ in Matlab:

```
x1 = Q\v;

reo = amd(Q);
R = chol(Q(reo,reo));
x2(reo) = R\(R'\v(reo));


x3(reo) = R\(v(reo)'/R)';
```

- For `x1`, Matlab automatically performs the reordering.
- Explicitly computing the reordering and Cholesky factor is needed when sampling GMRFs, and preferable if you will do many solves with Q for different v.

Morphological operations

Let A be a set of pixels in an image, and let S_{ij} be a structure element centered in pixel ij.

- Erosion of A: $A \ominus S = \{ij : S_{ij} \subset A\}$.
- Dilation of A: $A \oplus S = (A^c \ominus S)^c$, where A^c is the complement of the set A.
- Opening of A: $\psi_S(A) = (A \ominus S) \oplus S'$, where S' is S rotated 180 degrees.
- Closing of A: $\phi_S(A) = (A \oplus S) \ominus S'$.

Morphological operations on grayscale images

Let x be a grayscale image, and S a structure element. Then

- Erosion of x: $(x \ominus S)_{ij} = \min(x_{i'j'} : i'j' \in S_{ij}\}$.
- Dilation of x: $(x \oplus S)_{ij} = \max(x_{i'j'} : i'j' \in S_{ij}\}$.
- Opening of x: $\psi_S(x) = (A \ominus S) \oplus S'$.
- Closing of x: $\phi_S(x) = (A \oplus S) \ominus S'$.

Binary image

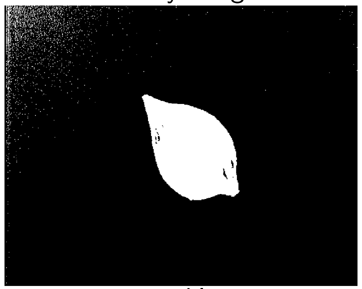


Image erosion

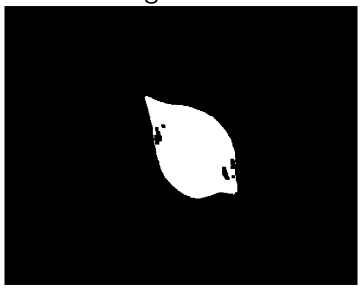


Image dilation

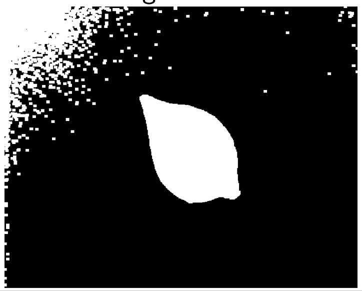
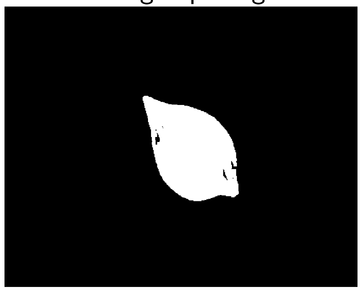


Image opening



Binary image

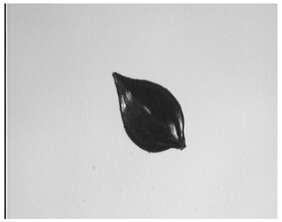


Image erosion

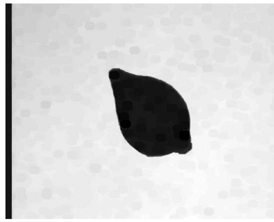


Image dilation

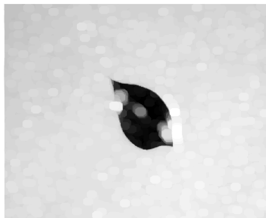
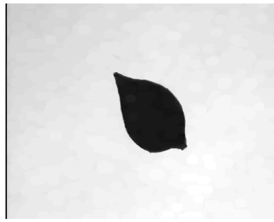


Image opening



Moment features

Let x be an iage of size $m \times n$. The moment of order (p,q) of x is

$$m_{pq} = \sum_{ij} i^p j^q x_{ij}$$

The $(0,0)$ moment, m_{00} is

- The area for binary images
- the sum of gray levels for grayscale images.

The image centroid is defined as

$$\left(\frac{m_{10}}{m_{00}}, \frac{m_{01}}{m_{00}}\right) := (\bar{i}, \bar{j})$$

Central moments:

$$\mu_{pq} = \sum_{ij} (i-\bar{i})^p (j-\bar{j})^q x_{ij}$$

Markov random field mixture models

- Hierarchical model for pixel values given classes:

$$\pi(\mathbf{Y}_i | z_i = k) \sim \mathsf{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

$$\pi(z_i) = \begin{cases} \pi_1 & \text{if } z_i = 1 \\ \pi_2 & \text{if } z_i = 2 \\ \vdots & \\ \pi_K & \text{if } z_i = K \end{cases}$$

- Assuming independence between the pixels is not realistic!
- In a Markov random field mixture model, we use the model

$$\pi(\mathbf{Y}_i | z_i = k) \sim \mathsf{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$
$$\mathbf{z} \sim \pi(\mathbf{z})$$

  here $\mathbf{z} = (z_1, \ldots, z_n)$ is a random field that takes values in $\{1, \ldots, K\}$, with density $\pi(\mathbf{z})$.
- Spatial dependencies modeled through $\pi(\mathbf{z})$.

Invariant Moments

- The central moments μ_{pq} are invariant to translations.
- The following quantity is invariant to both translations and scaling:

$$\eta_{pq} = \frac{\mu_{pq}}{\mu_{00}^{1+\frac{p+q}{2}}}$$

- The Hu-moments are also invariant to rotations. There are 8 such moments, the first two are

$$I_1 = \eta_{20} + \eta_{02}$$
$$I_2 = (\eta_{20} - \eta_{02})^2 + 4\eta_{11}^2$$

- Invariant moments are useful for image classification.

Constructing Markov random fields

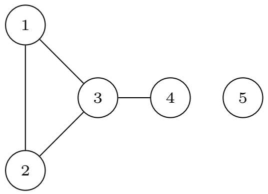
- How can we define a valid random field model for $\mathbf{z}$?
- Recall that we defined GMRFs using undirected graphs $\mathcal{G} = (E, V)$.
- Typically, we have the set of vertices V as the pixels in an image, and the set of edges E defines the dependence structure.
- We defined GMRFs using local constructions, such as the CAR models where we specified the joint distribution through the conditionals $\pi(x_i | x_{-i}) = \pi(x_i | x_{N_i})$.
- Today we will use local constructions to define discrete valued MRFs.
- Next lecture, we will look at parameter estimation and how to use the models for image segmentation.

Cliques

Definition

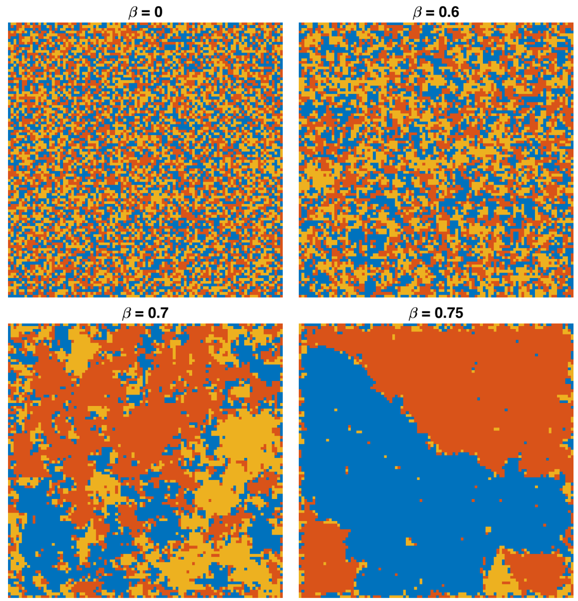
Let $\mathcal{G} = (V, E)$ be an undirected graph. A clique C of $\mathcal{G}$ is a subset of vertices such that every pair of vertices in C are adjacent.

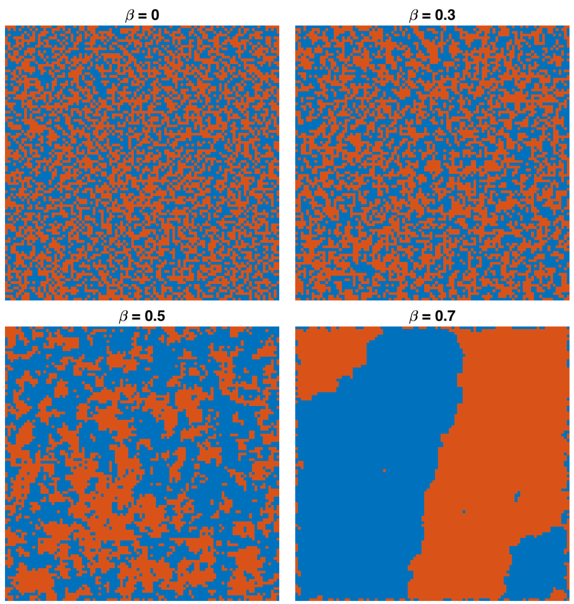
Example:



Cliques:

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$$
$$\{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}$$
$$\{1,2,3\}$$





Comments

The normalizing constant Z is given by summing all possible images x:

- With $K = 2$ and a 5×5 image, there are $2^{25} = 33554432$ possible images.
- We cannot compute Z for realistic images.

Sampling:

- Sampling $\pi(\mathbf{x})$ is difficult.
- Sampling $\pi(x_i|\mathbf{x}_{-i})$ is very easy. Can we use this?

Gibbs sampling

Assume that we have a distribution $\pi(\mathbf{x}) = \pi(x_1, \dots, x_n)$ that we want to sample from, where $\pi(x_i|x_{-i})$ is easy to sample.

Algorithm:

Step 1 Choose a starting value $\mathbf{x}^0$.

Step 2 Repeat for $i = 1, \dots, N$:

- Draw $x_1^{(i)}$ from $\pi(x_1|x_2^{(i-1)}, \dots, x_n^{(i-1)})$
- Draw $x_2^{(i)}$ from $\pi(x_2|x_1^{(i)}, x_3^{(i-1)}, \dots, x_n^{(i-1)})$

$$\vdots$$

- Draw $x_n^{(i)}$ from $\pi(x_n|x_1^{(i)}, \dots, x_{n-1}^{(i)})$

Step 3 Use $\mathbf{x}^{(K)}, \dots, \mathbf{x}^{(N)}$ as a sequence of dependent draws approximately from $\pi(\mathbf{x})$.

Under mild conditions, $\pi(\mathbf{x}^{(i)})$ converges to $\pi(\mathbf{x})$.

Chose K large enough so that the chain has converged.