Project assignment: Part 1
Image reconstruction using GMRFs
Spatial Statistics and Image Analysis, TMS016

A few years ago, the space probe Rosetta and its lander module Philae received much attention while performing a detailed study of comet 67P/Churyumov-Gerasimenko. On 12 November 2014, the mission performed the first soft landing on the comet and returned data from the surface. Fortunately, the equipment on the space probe worked well, and Rosetta has returned beautiful images of the comet. However, imagine a situation where there was something wrong with the camera equipment or the data transfer from a space probe, so that we only obtain partial or corrupted information. This scenario is not that unlikely: The Huygens spacecraft, that took images on Saturn's moon Titan ten years ago, had fatal communications issues resolved only when it was well on its way to Saturn.

On the course homepage you can find the images:

1. `titan.jpg`, taken by the Huygens spacecraft on Titan, see `http://antwrp.gsfc.nasa.gov/apod/ap050115.html`
2. `rosetta.jpg`, taken by the Rosetta spacecraft last year, see `www.esa.int/Our_Activities/Space_Science/Rosetta`

Start by loading one of the images. A good idea can be to start with the image of Titan, since that is smallest. Once you have the code working, you can then try the Rosetta image. Transform it to grayscale and convert the pixel values to double values between 0 and 1.

We will assume that the values of certain pixels in the image are lost in the data transfer from the spacecraft. For each pixel in the image, independently remove the pixel with a fixed probability p_c, and let $\mathbf{x}_o$ be the vector with with observed pixel values. Start with $p_c = 0.5$, so that approximately half of the pixels are observed. The task is now to reconstruct the missing pixels $\mathbf{x}_m$ given the observed values $\mathbf{x}_o$.

1. Use least-squares estimation (as in Computer Exercise 3) to estimate the parameters of a model $Y(s_i) = \mu + X(s_i)$, where $X(s)$ is a Gaussian field with a Matérn covariance function with $\nu = 1$. To reduce the computational cost, base the estimation on only a part of the observed pixels (say the first 10000). Plot the binned variogram estimate as well as the estimated Matérn variogram and report the parameter estimates.
2. The Gaussian field X can be approximated using a GMRF with precision matrix $\tau\mathbf{Q}$. Where $\mathbf{Q}$ is specified using the stencil

$$\kappa^4 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + 2\kappa^2 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & 4 & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 2 & -8 & 2 & \cdot \\ 1 & -8 & 20 & -8 & 1 \\ \cdot & 2 & -8 & 2 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}$$

   and $\tau = 2\pi/\sigma^2$. Compute the precision matrices (as in Computer Exercise 4) needed for reconstructing the missing pixels using this model (using the parameters estimated above) and reconstruct the missing values. Test changing the value of κ here to see how that affects the reconstruction, the least-squares estimate may not be the optimal value for the reconstruction.
3. How large can the amount of missing values be if we want to get a reasonable estimate of the true image? Try both images here and plot the reconstructions for some different percentages of missing pixels.