EXAMINATION: Tentamensskrivning i Matematisk Statistik (TMS061)

Time: Tuesday 29 May 2007

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Aid: You are allowed to use a scientific calculator and a half page (both sides) of hand written notes

Lab: Depending on the performance in the lab 0-5 points will be added to your test score to provide with your final score.

Grade: You need 42 points for 5, 30 points for 4 and 20 points for 3.

Motivate all your answers. Good Luck!

1) Determine the constant c so that the following function is a probability mass function: $f(x) = cx$ for $x = 1, 2, 3, 4$. (4p)

2) Let $X_1, X_2, \ldots, X_n$ be a random sample.

   a) What conditions do $X_1, X_2, \ldots, X_n$ have to satisfy? (1p)

   b) If $X_1, X_2, \ldots, X_n$ are additionally normally distributed $N(\mu, \sigma^2)$ what can you say about the distribution of $\bar{X}$? (2p)

   c) What is $E(\bar{X})$, $Var(\bar{X})$? (2p)

   d) Is $\bar{X}$ an unbiased estimator of the true mean? (1p)

3)
$$-0.19, 0.73, 2.18, -0.14, 0.11, 1.07, 0.04, -0.10, -0.83, 0.29$$
are 10 computer generated $N(0,1)$ observations. Test the hypothesis $\mu = 1$ with alternative $\mu \neq 1$ when

   a) $\sigma = 1$ is known (3p)

   b) σ is unknown (3p)

at significance level $\alpha = 0.1$.

4) Answer the following questions:

a) Can a null hypothesis be rejected at $\alpha = 0.01$ level when the p-value of the test was 0.007? (1p)

b) In an experiment if $P(A) = 0.6$ and $P(B) = 0.7$ is it possible that $P(A \cap B) = 0.2$? (2p)

c) Do X and $2 \cdot X$ have the same variance? (1p)

d) If $P(A) = 0.3, P(B) = 0.5$ and $P(A \cup B) = 0.65$, are the events A and B independent? (2p)

5) Consider the following frequency table:

Values	0	1	2	3	4	5
Observed frequency	75	140	108	66	9	2

Based on 400 observations is a binomial distribution $B(5, 0.3)$ an appropriate model? Perform a χ^2 test with $\alpha = 0.05$. (5p)

6) The performance of a mettalic device is to be tested. A sample of 14 specimen is taken to give $\bar{x} = 876.5$ and $s_x = 91.4$. After some adjustment has been performed a new sample of 16 specimen was taken to give $\bar{y} = 975.3$ and $s_y = 116.6$.

a) Formulate a suitable model to test the hypothesis that no change has occured with the introduction of the adjustment. (3p)

b) Provide with a confidence interval for the difference of the true means. (3p)

Use $\alpha = 0.05$.

7) The Rayleigh distribution has probability density function

$$f(x) = \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}, \quad x > 0, \quad 0 < \theta < \infty$$

Find the maximum likelihood estimator of θ. (5p)

8) Suppose that the random variable X denotes the number of arrivals in a bus station and is distributed as a Poisson random variable with mean 4 arrivals per hour. Compute the following probabilities:

a) $P(X = 0)$ (1p)

b) $P(X \geq 2)$ (1p)

c) $P(X \leq 4 | X \geq 2)$ (2p)

d) What is the probability that the first bus arrives after one hour? (3p)

9) Assume you are asked 4 questions to which you answer by guessing. For each correct answer you give, you win 1 point and for each wrong answer you loose 1 point. Let X be the maximum between your score and zero. What is the $E(X)$? (5p)

Solutions for TMS 061

1) $P(X=1)+P(X=2)+P(X=3)+P(X=4)=1 \Rightarrow$

$c+2c+3c+4c=1 \Rightarrow 10c=1 \Rightarrow c=1/10$

2) a) $X_1, \ldots, X_n$ have to be independent and identically distributed

b) $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$

$\bar{X} = \frac{X_1 + \ldots + X_n}{n}$ is a linear combination of normal r.vs $\Rightarrow$ is normal

c) $E\bar{X} = E\left(\frac{X_1+\ldots+X_n}{n}\right) = \frac{1}{n}\left(EX_1+\ldots+EX_n\right) =$

$= \frac{n \cdot \mu}{n} = \mu$

$Var\,\bar{X} = Var\left(\frac{X_1+\ldots+X_n}{n}\right) \overset{\text{indep.}}{=} \frac{Var X_1 + \ldots + Var X_n}{n^2}$

$= \frac{\sigma^2+\ldots+\sigma^2}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

d) $E\bar{X} = \mu \Rightarrow \bar{X}$ is unbiased estimator of μ.

3) $H_0: \mu = 1$ $\qquad \bar{x} = 0.316$

$H_1: \mu \neq 1$ $\qquad s = 0.835$

a) $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{0.316-1}{1/\sqrt{10}} = -2.163$

Reject H_0 if $z < -z_{\alpha/2} \Rightarrow -2.163 < -1.64$

so Reject H_0.

b) $T = \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t(n-1)$

$T = \frac{0.316 - 1}{0.835/\sqrt{10}} = -2.59$

$t_{0.05,9} = 2.821$

Reject H_0 if $T < -t_{0.05,9} = -2.59$

so Reject H_0.

4) a) Reject H_0 for any α-level so that $p < \alpha$. Since $p = 0.007 < 0.01$ so reject H_0.

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.7 - 0.2 = \underline{1.1}$

impossible

c) $Var(2 * X) = 4 \cdot Var X$ NO

d) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.3 + 0.5 - 0.65 = 0.15$

and $P(A)P(B) = 0.5 * 0.3 = 0.15 = P(A \cap B)$

yes.

5) $X \sim B(n,p) \Rightarrow P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

$p_1 = P(X=0) = 0.1681$ $E_1 = n p_1 = 67.228$

$p_2 = P(X=1) = 0.3601$ $E_2 = 144.06$

$p_3 = P(X=2) = 0.3087$ $E_3 = 123.48$

$p_4 = P(X=3) = 0.1323$ $E_4 = 52.92$

$p_5 = P(X=4) = 0.0283$ $E_5 = 11.34$

$p_6 = P(X=5) = 0.0024$ $E_6 = 0.9720 < 3$

combine them

$\tilde{P}_5 = 0.0283 + 0.0024$ $\qquad$ $\tilde{E}_5 = 42.28$

$$\chi_0^2 = \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i} = 6.3194 \quad , k-p-1 = 5-0-1 = 4 \text{ df}$$

$$\chi^2_{0.05,4} = 9.49$$

Since $\chi_0^2 = 6.3194 < 9.49$, we fail to reject H_0

6) Assume that $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$

a) $H_0: \mu_x = \mu_y$
$H_1: \mu_x \neq \mu_y$

$$S_p^2 = \frac{13 * 91.4^2 + 15 * 116.6^2}{14+16-2} = 11161.92 \Rightarrow S_p = 105.65$$

$$T = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{876.5 - 975.3}{105.65\sqrt{\frac{1}{14} + \frac{1}{16}}} = -2.56$$

$$t_{0.025,28} = 2.048$$

$T = -2.56 < -t_{0.025,28}$ so reject H_0.

b) $\mu_x - \mu_y \in \left[\bar{x} - \bar{y} \pm t_{0.025,28}\, S_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right] =$

$= [-177.98, -19.62]$

7) Let $X_1, X_2, \dots, X_n$ be a random sample.

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta} e^{-\frac{x_i^2}{2\theta}} =$$

$$= \frac{1}{\theta^n}\left(\prod_{i=1}^{n} x_i\right) e^{-\frac{\sum x_i^2}{2\theta}}$$

$$\ln L(\theta) = -n\ln\theta + \ln x_1 + \dots + \ln x_n - \frac{\sum x_i^2}{2\theta}$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = -\frac{n}{\theta} + \frac{\sum x_i^2}{2\theta^2} = 0 \Rightarrow \frac{\sum x_i^2}{2\theta} = n \Rightarrow$$

$$\boxed{\hat{\theta} = \frac{\sum x_i^2}{2n}}$$

8) a $P(X=0) = \frac{e^{-4} \cdot 4^0}{0!} = 0.0183$

b) $P(X \geq 2) = 1 - P(X<2) = 1 - P(X=0) - P(X=1) =$

$$= 1 - \frac{e^{-4} 4^0}{0!} - \frac{e^{-4} 4^1}{1!} = 0.9084$$

c) $P(X \leq 4 \mid X \geq 2) = \frac{P(X \geq 2 \text{ and } X \leq 4)}{P(X \geq 2)} =$

$$\frac{P(X = 2 \text{ or } 3 \text{ or } 4)}{P(X \geq 2)} = \frac{e^{-4} \cdot \left(4^2/2 + 4^3/6 + 4^4/24\right)}{0.9084} =$$

$$= \frac{0.5373}{0.9084} = 0.5914$$

d) Let T be the time for the first arrival

Then $T \sim Exp(4)$

$$P(T>1) = \int_1^{\infty} 4e^{-4x}\,dx = \cancel{4\frac{e^{-5x}}{-5}\Big|_1^{\infty}} = -e^{-4x}\Big|_1^{\infty} =$$

$$= \cancel{-\frac{4}{5}\left(e^{-\infty} - e^{-5}\right) = \frac{4}{5}e^{-5} = 0.0054} \quad = 0.018$$

g). $X = 0, 2, 4$

$$\binom{4}{3} = 4$$

$P(X=0) = P(4 \text{ wrong}) + P(3 \text{ wrong}, 1 \text{ correct}) +$

$+ P(2 \text{ wrong}, 2 \text{ correct}) = \frac{1}{2^4} + 4 \cdot \frac{1}{2^4} + 6\,\frac{1}{2^4} = 11/16$

$$\binom{4}{2} \overset{?}{=} 6$$

$P(X=2) = P(1 \text{ wrong}, 3 \text{ correct}) = 4/16$

$P(X=4) = P(4 \text{ correct}) = 1/16$

$EX = 0 \cdot \frac{11}{16} + 2 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16} = \frac{12}{16} = \underline{\underline{0{,}75}}$