Solution to exam

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Financial Time Series MSA410/TSM087

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Exam 1, reading period 14/15 LP4

Problem 1 (29 points)

Let

$$X_t - \frac{1}{2} X_{t-1} = Z_t, \quad t \in \mathbb{N}$$

where
- $Z_t \sim \mathcal{N}(0,1)$ $\forall t \in \mathbb{N}_0$ and $(Z_t, t \in \mathbb{N})$ independent
- $\mathbb{E}(X_0) = 0$, $\mathbb{E}(X_0^2) = \frac{4}{3}$

(2) a) A time series is a real-valued series of (1/2) observations $(x_t, t \in \mathbb{T})$ with respect to an index (1/2) set $\mathbb{T} \subset \mathbb{R}$.
A time series model for the observed data $(x_t, t \in \mathbb{T})$ is a specification of the joint distributions (1/2) of a sequence of random variables $(X_t, t \in \mathbb{T})$ of which $(x_t, t \in \mathbb{T})$ (1/2) postulates to be a realization.
(If no confusion arises, "time series" is used for both: the data and the process.)

(1 1/2) b) Let $X = (X_t, t \in \mathbb{Z})$ be a time series with (1/2) $\mathrm{Var}(X_t) < +\infty$ for all $t \in \mathbb{Z}$. The time series X is called (weakly) stationary if

(1/2) (i) $\exists \mu \in \mathbb{R} : \forall t \in \mathbb{Z} : \mathbb{E}(X_t) = \mu$ and

(1/2) (ii) $\forall r, s, h \in \mathbb{Z} : \gamma_X(r,s) = \gamma_X(r+h, s+h)$,
where $\gamma_X(r,s) := \mathrm{Cov}(X_r, X_s) = \mathbb{E}((X_r - \mathbb{E}(X_r))(X_s - \mathbb{E}(X_s)))$.

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c) Let X be a stationary time series. The autocovariance function (ACVF) $\gamma_X : \mathbb{Z} \to \mathbb{R}$ of X is defined by

$$\text{(1/2)} \quad \gamma_X(h) := \operatorname{Cov}(X_{t+h}, X_t), \quad h \in \mathbb{Z},$$

where $t \in \mathbb{Z}$ is chosen arbitrarily due to stationarity. (1/2)

($12\frac{1}{2}$)

d) Claim: X as given in the problem is weakly stationary.

Proof: The proof is divided into several steps.

(2) (i) Show that $\mathbb{E}(X_t) = \mathbb{E}(X_0) = 0$.

$$\mathbb{E}(X_t) \overset{\text{def.}}{=} \mathbb{E}\left(\tfrac{1}{2} X_{t-1} + Z_t\right) \overset{\text{lin.}}{\underset{(1/2)}{=}} \tfrac{1}{2}\mathbb{E}(X_{t-1}) + \overbrace{\mathbb{E}(Z_t)}^{=0}$$

$$\overset{(1/2)}{=} \tfrac{1}{2}\mathbb{E}(X_{t-1})$$

$$\overset{(1/2)}{=} \cdots = \left(\tfrac{1}{2}\right)^t \mathbb{E}(X_0)$$

$$\overset{(1/2)}{=} 0$$

$\Rightarrow$ Condition (i) of stationarity holds.

(3) (ii) Show that $\mathbb{E}(X_t^2) = \mathbb{E}(X_0^2) = \frac{4}{3}$.

Variant 1

$$\mathbb{E}(X_1^2) = \mathbb{E}\left(\left(\tfrac{1}{2}X_0 + Z_1\right)^2\right) \overset{\text{lin.}}{\underset{(1/2)}{=}} \tfrac{1}{4}\overbrace{\mathbb{E}(X_0^2)}^{=4/3} + 2\cdot\tfrac{1}{2}\mathbb{E}(X_0 Z_1) + \overbrace{\mathbb{E}(Z_1^2)}^{=1} \quad (1/2)$$

$$\overset{X_0 \perp\!\!\!\perp Z_1}{\underset{(1/2)}{=}} \tfrac{1}{4}\cdot\tfrac{4}{3} + \underbrace{\mathbb{E}(X_0)\,\mathbb{E}(Z_1)}_{=0} + 1$$

$$\overset{(1/2)}{=} \tfrac{4}{3}$$

recursively:

$$\mathbb{E}(X_t^2) = \mathbb{E}\left(\left(\tfrac{1}{2}X_{t-1} + Z_t\right)^2\right)$$

$$\overset{\text{see } X_1^2}{\underset{(1/2)}{=}} \tfrac{1}{4}\underbrace{\mathbb{E}(X_{t-1}^2)}_{\frac{4}{3}\text{ by recursion}} + 1 \quad (1/2)$$

$\rightarrow$ could be done by induction

$$= \tfrac{1}{3} + 1 = \tfrac{4}{3}$$

//

Variant 2

$$\mathbb{E}(X_t^2) = \mathbb{E}\left(\left(\tfrac{1}{2}X_{t-1} + Z_t\right)^2\right) \qquad X_{t-1} \perp\!\!\!\perp Z_t$$

$$\overset{\text{lin.}}{\underset{(1/2)}{=}} \tfrac{1}{4}\mathbb{E}(X_{t-1}^2) + 2\cdot\tfrac{1}{2}\overbrace{\mathbb{E}(X_{t-1}Z_t)}^{} + \overbrace{\mathbb{E}(Z_t^2)}^{=1}$$

$$\overset{(1/2)}{=} \tfrac{1}{4}\mathbb{E}(X_{t-1}^2) + \mathbb{E}(X_{t-1})\,\mathbb{E}(Z_t) + 1$$

$\overset{1/2}{=} \frac{1}{4}\,\mathbb{E}(X_{t-1}^2) + 1$

1/2 $\left\{ \begin{aligned} &\overset{\text{rec.}}{=} \frac{1}{4}\left(\frac{1}{4}\mathbb{E}(X_{t-2}^2)+1\right) + 1 \\ &= \frac{1}{4}\left(\frac{1}{4}\left(\frac{1}{4}\,\mathbb{E}(X_{t-3}^2)+1\right)+1\right)+1 \end{aligned}\right.$

$= \dots = \left(\frac{1}{4}\right)^t \mathbb{E}(X_0^2) + \sum_{j=0}^{t-1}\left(\frac{1}{4}\right)^j$

1/2 geom series

$= \left(\frac{1}{4}\right)^t \frac{4}{3} + \frac{1-\left(\frac{1}{4}\right)^t}{1-\frac{1}{4}}$

$= \cancel{\left(\frac{1}{4}\right)^t \frac{4}{3}} + \frac{4}{3}\left(1-\cancel{\left(\frac{1}{4}\right)^t}\right)$

1/2 $= \frac{4}{3}$ //

$\Rightarrow Cov(X_t, X_t) = Var(X_t) = \mathbb{E}(X_t^2) = \frac{4}{3} \quad \forall t \in \mathbb{N}$

3½ (iii) Show that $\mathbb{E}(X_{t-j}X_t) = \left(\frac{1}{2}\right)^j \frac{4}{3}$, $j \le t$.

$\mathbb{E}(X_{t-1}X_t) = \mathbb{E}\left(X_{t-1}\left(\frac{1}{2}X_{t-1} + Z_t\right)\right)$

1/2 $\overset{\text{lin.}}{=} \frac{1}{2}\,\mathbb{E}(X_{t-1}^2) + \mathbb{E}(X_{t-1}Z_t)$

1/2 $\overset{X_{t-1} \perp\!\!\!\perp Z_t}{=} \frac{1}{2}\,\mathbb{E}(X_{t-1}^2) + \underbrace{\mathbb{E}(X_{t-1})\,\mathbb{E}(Z_t)}_{=0}$

$= \frac{1}{2}\cdot\frac{4}{3} \overset{1/2}{=} \frac{2}{3}$

1/2 moreover: $\gamma_X(t-1, t) = \frac{1}{2}\gamma_X(t-1, t-1)$

and

$\gamma_X(t-j, t) = \mathbb{E}(X_{t-j}X_t) = \mathbb{E}\left(X_{t-j}\left(\frac{1}{2}X_{t-1} + Z_t\right)\right)$

1/2 $\overset{\text{see } t-1}{=} \frac{1}{2}\,\mathbb{E}(X_{t-j}X_{t-1})$

$= \frac{1}{2}\gamma_X(t-j, t-1)$

1/2 $\overset{\text{rec.}}{=} \dots = \left(\frac{1}{2}\right)^j \gamma_X(t-j, t-j)$

$= \left(\frac{1}{2}\right)^j \mathbb{E}(X_{t-j}^2)$

1/2 $= \left(\frac{1}{2}\right)^j \frac{4}{3}$

$\Rightarrow$ Condition (ii) of stationarity holds since γ_X just depends on the difference j and not on t.

details in text!

4 (iv) Condition (i) 1 and (ii) 1 hold as well as $Var(X_t) = \frac{4}{3} < +\infty$. 1

$\Rightarrow X$ is stationary

The autocovariance function is given by

(1) $\gamma_X(h) = \left(\frac{1}{2}\right)^{|h|}\frac{4}{3} = 2^{2-|h|}3^{-1} = \frac{2^{2-|h|}}{3} = \frac{1}{2^{|h|-2}3}$.

$\square$

(2)

e) The best linear predictor $P_n X_{n+h}$ for some $h>0$ of X_{n+h} with respect to $(X_t, t=1,..,n)$ is the linear combination

(1) $$P_n X_{n+h} = a_0 + \sum_{j=1}^{n} a_j X_{n+1-j},$$

where the coefficients $(a_j, j=0,..,n)$ are chosen such that

($\frac{1}{2}$) $$\mathbb{E}\left(\left(X_{n+h} - \left(a_0 + \sum_{j=1}^{n} a_j X_{n+1-j}\right)\right)^2\right)$$

($\frac{1}{2}$) is minimized.

($3\frac{1}{2}$)

f) Claim: The best linear predictor $P_2 X_3$ of the given stationary time series is

$$P_2 X_3 = \frac{1}{2} X_2$$

Proof: Prop. 1.3.13 from the lecture states that a_1, a_2 is given by the solution to

(1) $$\begin{cases} (\gamma_X(i-j))_{i,j=1}^{2} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \gamma_X(1) \\ \gamma_X(2) \end{pmatrix} \\ \text{and} \quad a_0 = \mathbb{E}(X_0)\left(1 - \sum_{i=1}^{2} a_i\right) = 0 \end{cases}$$

So $$\frac{4}{3}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{4}{3}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$$ (1)

$$\iff \quad a_1 + \frac{1}{2}a_2 = \frac{1}{2} \quad (i)$$
$$\frac{1}{2}a_1 + a_2 = \frac{1}{4} \quad (ii)$$

(1) $$\begin{cases} \overset{2(ii)-(i)}{\Longrightarrow} \quad \frac{3}{2} a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = 0} \\ \overset{\text{in }(i)}{\Longrightarrow} \quad \boxed{a_1 = \frac{1}{2}} \end{cases}$$

($\frac{1}{2}$) $$\Longrightarrow P_2 X_3 = a_0 + a_1 X_2 + a_2 X_1 = \frac{1}{2} X_2.$$

$\square$

(2½) g) The mean squared error of $P_n X_{n+h}$ is given by

(1) $MSE = \mathbb{E}((X_{n+h} - P_n X_{n+h})^2)$.

Claim: The MSE of $P_2 X_3$ is 1.

Proof:

$$\begin{aligned} MSE &\overset{(1/2)}{=} \mathbb{E}((X_3 - P_2 X_3)^2) \\ &\overset{(f)}{=} \mathbb{E}((X_3 - \tfrac{1}{2}X_2)^2) \\ (1/2)\ &= \mathbb{E}((\tfrac{1}{2}X_2 + Z_3 - \cancel{\tfrac{1}{2}X_2})^2) \\ &= \mathbb{E}(Z_3^2) \\ (1/2)\ &\overset{def}{=} 1 \end{aligned}$$

□

(4) h) Let x_2 be an observation of X_2, choose $M > 0$.

for $m = 1, \dots, M$ do (1)

(i) generate $z \sim \mathcal{N}(0,1)$ (1)

(ii) set $x_3(m) := \frac{1}{2}x_2 + z$ (1)

end

$x_3 := \frac{1}{M} \sum_{m=1}^{M} x_3(m)$ (1)

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Problem 2 (15 points)

(2) a) A time series X is called an ARMA(p,q) process if X is (1/2) stationary and if for all $t \in \mathbb{Z}$

$$(1/2)\ X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j},$$

where (1/2) $Z \sim WN(0, \sigma^2)$ and the polynomials

(1/2) $(1 - \sum_{j=1}^{p} \phi_j z^j)$ and $(1 + \sum_{j=1}^{q} \theta_j z^j)$ have no common factors.

(2) b) Definition: An ARMA(p,q) process X is causal if there exists a sequence of constants $(\psi_j, j \in \mathbb{N}_0)$ such that

(1/2) $\sum_{j=0}^{\infty} |\psi_j| < +\infty$ and

$$(\tfrac{1}{2}) \quad X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad , \quad t \in \mathbb{Z},$$

i.e, X is a MA(∞) process.

(1) Lemma 1: X is causal $\iff$ $\forall z \in \mathbb{C}$ s.t. $|z| \leq 1$:

$$1 - \sum_{j=1}^{p} \phi_j z^j \neq 0$$

(2) c) Definition: An ARMA(p,q) process is invertible if there exists a sequence $(\pi_j, j \in \mathbb{N}_0)$ s.t. $\sum_{j=0}^{\infty} |\pi_j| < +\infty$ and (1/2)

$$(\tfrac{1}{2}) \quad Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \quad , \quad t \in \mathbb{Z},$$

i.e, X is an AR(∞) process

(1) Lemma 2: X is invertible $\iff$ $\forall z \in \mathbb{C}$ s.t. $|z| \leq 1$:

$$1 + \sum_{j=1}^{q} \theta_j z^j \neq 0$$

(3) d) $(Z_t, t \in \mathbb{Z})$ independent and $Z_t \sim \mathcal{N}(0,1)$.

Consider three ARMA processes:

(3) (i) Claim: The AR(2) process X given by

$$X_t + 0.2 X_{t-1} - 0.48 X_{t-2} = Z_t$$

is invertible and causal.

Proof:

X is invertible by definition:

Set $\pi_0 = 1$, $\pi_1 := 0.2$, $\pi_2 := -0.48$, $\pi_j := 0$ for $j > 2$, (1/2)

then $\sum_{j=0}^{\infty} |\pi_j| = 1.68 < +\infty$ and (1/2)

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \quad , t \in \mathbb{Z}.$$

Observe that

$$1 + 0.2z - 0.48z^2 = (1 - 0.6z)(1 + 0.8z) \quad (\tfrac{1}{2})$$

$$\Rightarrow \quad 1 + 0.2z - 0.48z^2 \overset{!}{=} 0 \quad \overset{(1/2)}{\iff} \quad z = \frac{1}{0.6} \text{ or } z = -\frac{1}{0.8}$$

$$\overset{(1/2)}{\Rightarrow} \quad 1 + 0.2z - 0.48z^2 \neq 0 \quad \forall z \in \mathbb{C} \text{ s.t. } |z| \leq 1$$

$\overset{\text{Lemma 1}}{\Longrightarrow}$ (1/2) X is causal. □

3 (ii) Claim: The ARMA(1,1) process given by

$$X_t + 0.6 X_{t-1} = Z_t - 1.2 Z_{t-1}$$

is causal but not investible.

Proof: $1 + 0.6 z \overset{!}{=} 0$ (1/2) $\Longleftrightarrow$ (1/2) $z = -\frac{1}{0.6}$, so $|z| > 1$

(1/2) $\overset{\text{Lemma 1}}{\Longrightarrow}$ X is causal.

$1 - 1.2 z \overset{!}{=} 0$ (1/2) $\Longleftrightarrow$ (1/2) $z = \frac{1}{1.2}$, so $|z| \leq 1$

(1/2) $\overset{\text{Lemma 2}}{\Longrightarrow}$ X is not investible. □

3 (iii) Claim: The ARMA(1,2) process given by

$$X_t + 1.6 X_{t-1} = Z_t - 0.4 Z_{t-1} + 0.04 Z_{t-2}$$

is not causal but investible.

Proof:

$1 + 1.6 z \overset{!}{=} 0$ (1/2) $\Longleftrightarrow$ (1/2) $z = -\frac{1}{1.6}$, so $|z| \leq 1$

(1/2) $\overset{\text{Lemma 1}}{\Longrightarrow}$ X is not causal

$1 - 0.4 z + 0.04 z^2$ (1/2) $= (1 - 0.2 z)^2 \overset{!}{=} 0$ (1/2) $\Longleftrightarrow$ $z = \frac{1}{0.2}$, so $|z| > 1$

(1/2) $\overset{\text{Lemma 2}}{\Longrightarrow}$ X is investible □

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Problem 3 (16 points)

Let X be a time series.

1 a) Let $n \in \mathbb{N}$ be fixed. The sample mean $\overline{X}_n$ of $(X_t, t = 1, \dots, n)$ is defined by

$$(1) \quad \overline{X}_n := \frac{1}{n} \sum_{t=1}^{n} X_t .$$

3 b) For an iid sequence $(X_t, t \in \mathbb{N})$, the sample mean is an estimator for the expectation (1), i.e., $\mathbb{E}(X_1) \approx \overline{X}_n$,

which is due to the law of large numbers.

② Claim: $\overline{X}_n$ is an unbiased estimator of $\mathbb{E}(X_1)$, i.e.,

$$\mathbb{E}(\overline{X}_n) = \mathbb{E}(X_1).$$

Proof:

$$\mathbb{E}(\overline{X}_n) \overset{1/2}{=} \mathbb{E}\left(\frac{1}{n}\sum_{t=1}^{n} X_t\right)$$

$$\overset{1/2 \text{ lin.}}{=} \frac{1}{n}\sum_{t=1}^{n} \mathbb{E}(X_t)$$

$$\overset{1/2 \text{ identically}}{=} \frac{1}{n}\sum_{t=1}^{n} \mathbb{E}(X_1)$$

$$= \frac{1}{n} n \cdot \mathbb{E}(X_1) \overset{1/2}{=} \mathbb{E}(X_1)$$

$\square$

④ c) The classical decomposition model is given by the decomposition of the time series into

① $$X_t = m_t + s_t + y_t$$

where

① • $m: \mathbb{Z} \to \mathbb{R}$ is a slowly changing function, the trend component,

① • $s: \mathbb{Z} \to \mathbb{R}$ is a function with known period d, i.e., $s_{t+d} = s_t \;\forall t \in \mathbb{Z}$ and $\sum_{j=1}^{d} s_j = 0$, called the seasonal component,

① • $(y_t, t \in \mathbb{Z})$ is a stationary stochastic process.

④ d) Variant 1

① "S1": estimation by moving averages

① • Set $\hat{m}_t$ as moving average depending on d even or odd

① • Average over trend-eliminated series elements with the same seasonal component, called w_k

① • Rescale the w_k's to satisfy $\sum_{j=1}^{d} s_j = 0$.

Variant 2

(1) "S2": elimination by differencing

(1) • Define that lag-d differencing operator ∇_d by

$$\nabla_d X_t = X_t - X_{t-d}$$

(1) • Observe that

$$\nabla_d X_t = \nabla_d m_t + \nabla_d y_t$$

(1) • $\nabla_d X_t$ is a stochastic process without seasonal component.

(4) e) Definition: A stochastic process $X = (X_t, t \in \mathbb{Z})$ is called a GARCH(p,q) process if it is (1/2) stationary and if

(1) $X_t = \sigma_t Z_t$,

where

(1/2) • $Z \sim IID\ \mathcal{N}(0,1)$,

(1/2) • $\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2$,

▷ $\alpha_0 > 0$, $\alpha_j \geq 0$, $j = 1, .., p$

▷ $\beta_i \geq 0$, $i = 1, .., q$

(1/2) • $Z_t \perp\!\!\!\perp (X_{t-j}, j \in \mathbb{N}) \quad \forall t \in \mathbb{Z}$.

(1) Lemma X GARCH(p,q) process

$$\Rightarrow X^2 := (X_t^2, t \in \mathbb{Z}) \text{ ARMA}(\max\{p,q\}, q) \text{ process.}$$

Π $X_t^2 = \alpha_0 + \sum_{i=1}^{\max\{p,q\}} (\alpha_i + \beta_i) X_{t-i}^2 + \eta_t - \sum_{j=1}^{q} \beta_j \eta_{t-j}$,

where $\eta_t := X_t^2 - \sigma_t^2$.

(16)