Stochastic differential equations are frequently used in finance to model prices. Assume that you are given data and you were told that the data can be modeled by the stochastic differential equation

$$X_t = X_{t_0} + \int_{t_0}^{t} \mu X_s \, \mathrm{d}s + \int_{t_0}^{t} \sqrt{\delta + \sigma X_s^2} \, \mathrm{d}B_s,$$

where $t > t_0$ with $t_0 \in \mathbb{Z}$, μ, δ, and σ are real numbers, $(B_t, t \in \mathbb{R})$ is a Brownian motion, and X_{t_0} is the solution at time t_0. Furthermore, you are told that a good approximation of the solution can be found with the *Euler–Maruyama scheme* given in recursive form by

$$X_t = (1 + \mu) X_{t-1} + \sqrt{\delta + \sigma X_{t-1}^2} \, Z_t, \tag{1}$$

where the stochastic process $Z := (Z_t, t \in \mathbb{Z})$ is IID$(0, 1)$ noise such that $Z_t \sim \mathcal{N}(0, 1)$ for all t and it models the increments of the Brownian motion. We are allowed to assume that X_t is independent of Z_{t+h} for all $t \in \mathbb{Z}$ and all $h > 0$.

The goal of this exam is to fit an appropriate time series model from the lecture to compute errors and to derive predictors of the future development of the price modeled by the stochastic differential equation. This task is split into the following problems to be solved by you.

Assume throughout the exam that $\mu = -1$, $\delta = 1$, and $\sigma = 1/2$. Furthermore you were told that $\mathsf{Var}[X_t] = 2$ for all $t \in \mathbb{Z}$.

Problem 1

(a) Show that $X = (X_t, t \in \mathbb{Z})$ is an ARCH(1) process.

(b) Show that X is white noise and compute the parameters.

(c) Is X IID noise? Prove your claim. (*Hint:* You might want to reconsider this task after solving Problem 2.)

$(9 + 3 + 4 = 16$ points$)$

Problem 2

(a) Compute the mean of X^2.

(b) Assume that $\mathbb{E}[X_t^4]$ does not depend on t. Compute $\mathbb{E}[X_t^4]$. (*Hint:* If $Y \sim \mathcal{N}(0, \sigma^2)$, then $\mathbb{E}[Y^4] = 3\sigma^4$.)

(c) Compute $\mathsf{Cov}[X_t^2, X_{t+h}^2]$ for all $t, h \in \mathbb{Z}$.

(d) Conclude that X^2 is stationary.

(e) Show that $\tilde{Z} := (\tilde{Z}_t, t \in \mathbb{N})$ with $\tilde{Z}_t := X_t^2 - (\delta + \sigma X_{t-1}^2)$ is white noise with mean zero and variance 24.

(f) Show that X^2 is an AR(1) process with mean 2 and $\phi_1 = 1/2$.

(g) Assume that you are given (X_1^2, X_2^2). Compute the best linear predictor of X_3^2. (*Hint:* All relevant parameters are given in the previous exercises even if you did not solve them.)

(h) Compute the mean squared error of the best linear predictor.

(i) Furthermore you received random numbers for Z_3, which are denoted by $Z_3^{(1)}$ and $Z_3^{(2)}$. Compute two different predictors of X_3^2 based on the parametric bootstrap algorithm using (1).

(j) Compute the theoretical mean squared errors for both predictors in (i) in the same way as you computed it in (h) for the best linear predictor.

(k) Compare the two estimators of the parametric bootstrap and the best linear predictor obtained before. Which predictor would you recommend and why?

$(1 + 3.5 + 7.5 + 1.5 + 5.5 + 4 + 5.5 + 2 + 5 + 5.5 + 3 = 44 \text{ points})$