FTS16-1-1

Financial Time Series 2016-06-03

Exam 1, Läsår 15/16, LP4

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Problem 1

[12] (a) • Since X is assumed to be stationary, the mean of X is $\mathbb{E}(X_t)$ (1/2) for arbitrary $t \in \mathbb{Z}$, which is constant (1/2) by stationarity.

$$\mathbb{E}(X_t) \overset{def}{=} \mathbb{E}(\phi X_{t-1} + Z_t)$$
$$\overset{lin.\,\mathbb{E}}{=} \phi\, \mathbb{E}(X_{t-1}) + \mathbb{E}(Z_t) \quad (1/2)$$
$$\overset{\substack{stat\\ Z\sim WN(0,\sigma^2)}}{=} \phi\, \mathbb{E}(X_t) + 0 \quad (1/2)$$

$$\iff (1-\phi)\,\mathbb{E}(X_t) = 0$$
$$\iff \mathbb{E}(X_t) = 0 \quad (1/2) \text{ since } (1-\phi) \neq 0 \ (1/2)$$

• Since X is stationary, (1/2) the variance is constant (1/2) in time and therefore $Var(X_t)$ for any $t \in \mathbb{Z}$.

$$Var(X_t) \overset{(1/2)}{=} \mathbb{E}(X_t^2) - \mathbb{E}(X_t)^2$$
$$\overset{mean}{=} \mathbb{E}(X_t^2) - 0$$
$$\overset{def}{=} \mathbb{E}\left((\phi X_{t-1} + Z_t)^2\right)$$
$$\overset{(1/2)}{=} \mathbb{E}\left(\phi^2 X_{t-1}^2 + Z_t^2 + 2\phi X_{t-1} Z_t\right)$$
$$\overset{lin}{=} \phi^2 \underbrace{\mathbb{E}(X_{t-1}^2)}_{\overset{mean\,0}{=} Var(X_{t-1})\ (1/2)} + \underbrace{\mathbb{E}(Z_t^2)}_{\substack{\overset{WN(0,\sigma^2)}{=} Var(Z_t^2) \\ = \sigma^2\ (1/2)}} + 2\phi \underbrace{\mathbb{E}(X_{t-1} Z_t)}_{\substack{\overset{\substack{Z\sim WN(0,\sigma^2)\\ Z_t \text{ uncor. } X_{t-1}}}{=} \mathbb{E}(X_{t-1})\mathbb{E}(Z_t) \\ = 0 \cdot 0\ (1/2)}}$$
$$(1/2) \overset{stat.}{=} \phi^2 Var(X_t) + \sigma^2$$

$$\iff (1-\phi^2)\,Var(X_t) = \sigma^2 \quad \overset{|\phi| \neq 1\ (1/2)}{\iff} Var(X_t) = \frac{\sigma^2}{1-\phi^2} \quad (1/2)$$

• Since X is stationary, the autocovariance function just depends on the distance h and is given by $^{(1/2)}$ $Cov(X_t, X_{t+h})$ for any$^{(1/2)}$ $t \in \mathbb{Z}$. Wlog let $h > 0$.

$$
\begin{aligned}
Cov(X_t, X_{t+h}) &\overset{(1/2)}{=} \mathbb{E}(X_t X_{t+h}) - \overbrace{\mathbb{E}(X_t)}^{=0}\mathbb{E}(X_{t+h}) \\
&= \mathbb{E}(X_t \cdot (\phi X_{t+h-1} + Z_{t+h})) \\
(1/2) &\overset{lin}{=} \phi \mathbb{E}(X_t \cdot X_{t+h-1}) + \mathbb{E}(X_t Z_{t+h}) \\
&\overset{Z \sim WN(0,\sigma^2)}{=} \phi\, Cov(X_t, X_{t+h-1}) + \mathbb{E}(X_t)\underbrace{\mathbb{E}(Z_{t+h})}_{=0} \\
(1/2) &= \phi\, Cov(X_t, X_{t+h-1}) \\
&\overset{rec}{=} \phi \cdot \phi\, Cov(X_t, X_{t+h-2}) \\
&\overset{rec}{=} \dots \\
(1/2) &= \phi^h\, Cov(X_t, X_t) \\
&= \phi^h\, Var(X_t) \\
(1/2) &\overset{var}{=} \phi^h \frac{\sigma^2}{1-\phi^2}
\end{aligned}
$$

$$\Rightarrow \forall h \in \mathbb{Z} \quad Cov(X_t, X_{t+h}) = \frac{\phi^{|h|}\sigma^2}{1-\phi^2} =: \gamma_X(h) \quad (1/2)$$

$\boxed{5}$

(b) An ARMA(p,q) process X is causal if there exists a real-valued sequence $(\psi_j, j \in \mathbb{N})$ s.t. $\sum_{j=0}^{\infty} |\psi_j| < +\infty$ $^{(1/2)}$ and

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z} \quad (1/2)$$

Furthermore

Lemma (1): X is causal $\Leftrightarrow$ $\forall z \in \mathbb{C}$ s.t. $|z| \le 1$:

$$1 - \sum_{j=1}^{p} \phi_j z^j \neq 0,$$

where the ARMA(p,q) process is given by $(1/2)$

$$X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j}$$

X is an AR(1), i.e., an ARMA(1,0) process. (1/2)

By the lemma consider

$$1 - \phi z = 0 \quad \left(\tfrac{1}{2}\right)$$

$$\iff \phi z = 1$$

$$\overset{\phi \neq 0}{\iff} z = \frac{1}{\phi} \quad \left(\tfrac{1}{2}\right)$$

$\Rightarrow$ X is causal iff $|\phi| < 1$ (1)

[4] (c) An ARMA(p,q) process is invertible if there exists a real-valued sequence $(\pi_j, j \in \mathbb{N}_0)$ s.t. $\sum_{j=0}^{\infty} |\pi_j| < +\infty$ $\left(\tfrac{1}{2}\right)$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}, \quad t \in \mathbb{Z} \quad \left(\tfrac{1}{2}\right)$$

Since the considered X is an AR(1) = ARMA(1,0) process, $\left(\tfrac{1}{2}\right)$

set $\pi_0 = 1$, $\left(\tfrac{1}{2}\right)$

$\pi_1 = -\phi$ $\left(\tfrac{1}{2}\right)$

Then
$$Z_t = X_t - \phi X_{t-1}$$
$$= \pi_0 X_t + \pi_1 X_{t-1} \quad \left(\tfrac{1}{2}\right)$$

and

$$|\pi_0| + |\pi_1| = 1 + |\phi| < +\infty \quad \text{for } |\phi| < +\infty \quad \left(\tfrac{1}{2}\right)$$

Therefore X is invertible for all finite ϕ. $\left(\tfrac{1}{2}\right)$

[15] (d) The maximum likelihood estimators of ϕ and σ^2 are determined from the expression (by the lecture)

$$\hat{\sigma}^2 = \frac{1}{n} S(\hat{\phi}), \quad \left(\tfrac{1}{2}\right)$$

where

$$S(\hat{\phi}) = \sum_{j=1}^{n} \frac{1}{r_{j-1}} (X_j - \hat{X}_j)^2 \quad \left(\tfrac{1}{2}\right)$$

and $\hat{\phi}$ minimizes $\left(\tfrac{1}{2}\right)$

$$\ell(\phi) = \ln\left(\frac{1}{n} S(\phi)\right) + \frac{1}{n} \sum_{j=1}^{n} \ln r_{j-1}. \quad \left(\tfrac{1}{2}\right)$$

Here n is the number of observations, $\hat{X}_1 := 0$, $\hat{X}_j = \mathbb{E}(X_j | X_1, \ldots X_{j-1})$ $\left(\tfrac{1}{2}\right)$

and $r_j := \dfrac{\mathrm{Var}(X_{j+1} - \hat{X}_{j+1})}{\sigma^2}$ $\left(\tfrac{1}{2}\right)$

For X we have

$$\underline{\underline{r_0}} = \frac{Var(X_1-\hat{X}_1)}{\sigma^2} \overset{(\frac{1}{2})}{=} \frac{Var(X_1)}{\sigma^2} \overset{(4)}{=} \frac{1}{\sigma^2}\,\frac{\sigma^2}{1-\phi^2} = \underline{\underline{\frac{1}{1-\phi^2}}} \quad (\tfrac{1}{2})$$

and

$$\begin{aligned}
\underline{\hat{X}_2} &= E(X_2 \mid X_1) \\
&\overset{\text{def.}}{=} E(\phi X_1 + z_2 \mid X_1) \quad (\tfrac{1}{2}) \\
&\overset{\text{lin.}}{=} \phi E(X_1 \mid X_1) + E(z_2 \mid X_1) \quad (\tfrac{1}{2}) \\
&\overset{z\sim WN(0,\sigma^2)}{=} \phi X_1 + E(z_2) \quad (\tfrac{1}{2}) \\
&= \phi X_1 \quad (\tfrac{1}{2})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \underline{\underline{r_1}} &= \frac{Var(X_2-\hat{X}_2)}{\sigma^2} \overset{(\frac{1}{2})}{=} \frac{1}{\sigma^2} Var(\phi X_1 + z_2 - \phi X_1) \\
&\overset{(\frac{1}{2})}{=} \frac{1}{\sigma^2} Var(z_2) \\
&\overset{z\sim WN(0,\sigma^2)}{=} \frac{1}{\sigma^2}\cdot\sigma^2 = \underline{\underline{1}} \quad (\tfrac{1}{2})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow S(\phi) &\overset{(\frac{1}{2})}{=} \frac{1}{r_0}(X_1-\hat{X}_1)^2 + \frac{1}{r_1}(X_2-\hat{X}_2)^2 \\
&\overset{(\frac{1}{2})}{=} (1-\phi^2)X_1^2 + 1\cdot(X_2-\phi X_1)^2 \\
&= X_1^2 - \cancel{\phi^2 X_1^2} + X_2^2 + \cancel{\phi^2 X_1^2} - 2X_2\phi X_1 \\
&\overset{(\frac{1}{2})}{=} X_1^2 + X_2^2 - 2\phi X_1 X_2
\end{aligned}$$

and

$$l(\phi) \overset{(\frac{1}{2})}{=} \ln\left(\frac{1}{2}(X_1^2+X_2^2-2\phi X_1X_2)\right) + \frac{1}{2}\left(\ln\frac{1}{1-\phi^2} + \ln 1\right)$$

To minimize wrt ϕ consider the observations x_1 and x_2 and

$$\begin{aligned}
\frac{\partial}{\partial\phi} l(\phi) &\overset{(\frac{1}{2})}{=} \frac{-2x_1x_2}{x_1^2+x_2^2-2\phi x_1x_2} + \cancel{\frac{1}{2}}\,\frac{-\cancel{2}\phi}{1-\phi^2} \\
&= \frac{\phi}{1-\phi^2} - \frac{2x_1x_2}{x_1^2+x_2^2-2\phi x_1x_2} \overset{!}{=} 0
\end{aligned}$$

$$x_1^2+x_2^2-2\phi x_1x_2 \overset{!}{=} 0 \iff \phi = \frac{x_1^2+x_2^2}{2x_1x_2} = \frac{(x_1-x_2)^2}{2x_1x_2}+1 = \frac{(x_1+x_2)^2}{2x_1x_2}-1 \Rightarrow |\phi|>1$$

$$|\phi|\neq 1 \ (\tfrac{1}{2}) \iff \phi = (1-\phi^2)\frac{2x_1x_2}{x_1^2+x_2^2-2\phi x_1x_2}$$

$$\text{causal, } |\phi|<1 \iff (\tfrac{1}{2}) \quad \phi(x_1^2+x_2^2-2\phi x_1x_2) = 2x_1x_2 - 2\phi^2 x_1x_2 \quad (\tfrac{1}{2})$$

$$\Longleftrightarrow \quad \hat{\phi} = \frac{2x_1x_2}{x_1^2+x_2^2} \quad (1/2)$$

Observe that $\lim_{\phi \downarrow -1} \ell(\phi) = \ln(\frac{1}{2}(X_1+X_2)^2) + \frac{1}{2}(\ln 1 - \overbrace{\ln(1-\phi^2)}^{\to -\infty}) = +\infty$ (1/2)

and $\lim_{\phi \uparrow 1} \ell(\phi) = \ln(\frac{1}{2}(X_1-X_2)^2) + \frac{1}{2}(\ln 1 + \infty) = +\infty$, therefore we found a minimum. (1/2)

$$\Rightarrow \quad \hat{\phi} := \frac{2x_1x_2}{x_1^2+x_2^2} \quad (1/2)$$

and
$$\hat{\sigma}_2 = \frac{1}{2} S(\hat{\phi}) \overset{(1/2)}{=} \frac{1}{2}\left(x_1^2+x_2^2 - 2\cdot\frac{2x_1x_2}{x_1^2+x_2^2}x_1x_2\right)$$
$$\left.\begin{aligned} &= \frac{1}{2}\frac{1}{x_1^2+x_2^2}\left((x_1^2+x_2^2)^2 - 4x_1^2x_2^2\right) \\ &= \frac{1}{2(x_1^2+x_2^2)}\left(x_1^4 + \cancel{2}x_1^2x_2^2 + x_2^4 - \cancel{4}^{2}x_1^2x_2^2\right)\end{aligned}\right\} (1/2)$$
$$= \frac{(x_1^2-x_2^2)^2}{2(x_1^2+x_2^2)} \quad (1/2)$$

$\boxed{5}$ (c) Recall that the best linear predictor $b^{\ell}_{n+h}(X^n)$ for some $h>0$ of X_{n+h} wrt $X^n = (X_1, \dots, X_n)$ is the linear combination

$$b^{\ell}_{n+h}(X^n) = a_0 + \sum_{j=1}^{n} a_j X_{n+1-j} \quad (1/2)$$

such that $MSE(X_{n+h}, b^{\ell}_{n+h}(X^n))$ is minimized (1/2)

From the lecture we know that the coefficients (a_0, a_1, a_2) are given by the solution of

$$(\gamma_X(i-j))_{i,j=1}^{2}\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}\gamma_X(1)\\ \gamma_X(2)\end{pmatrix} \quad (1/2)$$

and
$$a_0 = E(X_0)\left(1 - \sum_{i=1}^{2} a_i\right) = 0 \quad (1/2)$$

So from (a)

$$\frac{\hat{\sigma}^2}{1-\hat{\phi}^2}\begin{pmatrix}1 & \hat{\phi}\\ \hat{\phi} & 1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = \frac{\hat{\sigma}^2}{1-\hat{\phi}^2}\begin{pmatrix}\hat{\phi}\\ \hat{\phi}^2\end{pmatrix} \quad (1/2)$$

$$\Longleftrightarrow \quad \begin{cases} a_1 + \hat{\phi} a_2 = \hat{\phi} & \text{(i)} \\ \hat{\phi} a_1 + a_2 = \hat{\phi}^2 & \text{(ii)} \end{cases} \quad (1/2)$$

$$\overset{\phi(i)-(ii)}{\Longrightarrow} \quad (\hat{\phi}^2 - 1)a_2 = 0 \quad \overset{|\hat{\phi}| \neq 1 \text{ since } |x_1| \neq |x_2|}{\Longleftrightarrow} \quad a_2 = 0 \quad (1/2)$$

$$\Rightarrow \quad a_1 = \hat{\phi} \quad (1/2)$$

$$\Rightarrow \quad b_3((x_1,x_2)) = \hat{\phi}\cdot x_2 = \frac{2x_1x_2^2}{x_1^2+x_2^2} \quad 1$$

$\frac{1}{2}$ $\phi_{MLE}/\hat{\phi}$

$\frac{1}{2}$ numbers from d

$\boxed{3}$

(f) The mean squared error is according to the lecture

$$MSE(b_3^\ell(X_1,X_2),X_3) \overset{\frac{1}{2}}{=} \gamma_X(0) - (a_1;a_2)\begin{pmatrix}\gamma(1)\\ \gamma(2)\end{pmatrix}$$

alt: by hand from definition

$$\overset{\frac{1}{2}}{=} \frac{\hat{\sigma}^2}{1-\hat{\phi}^2} - \hat{\phi}\frac{\hat{\sigma}^2}{1-\hat{\phi}^2}\cdot\hat{\phi}$$

$$= \frac{\hat{\sigma}^2}{1-\hat{\phi}^2}(1-\hat{\phi}^2)$$

$$\overset{\frac{1}{2}}{=} \hat{\sigma}^2 = \frac{(x_1^2-x_2^2)^2}{2(x_1^2+x_2^2)} \quad \frac{1}{2}$$

which does not include the model fitting error of $\hat{\phi}$ and $\hat{\sigma}^2$ 1

Problem 2

- Testing data

  3 out of: iid, seasonality, stationarity, (non) linearity, (G)ARCH effects, background of data, ~~ACVF~~, stochastic variance, plot / visualize

  + 1 algorithm

- Model fitting

  3 out of: order selection (ARMA/GARCH), parameter estimation (ARMA/GARCH), nonparametric methods

  + 1 algorithm

- Forecasting

  3 out of: best (linear) predictors, ARMA forecasting, parametric bootstrap, one-step-ahead, iid (distribution)

  + 1 algorithm

- Error analysis

  4 out of: evaluation / prediction subsample (1), MSE (1), nonlinear method (1), confidence intervals