Financial Time Series

MSA410 / TMS088

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(1) $X_t = (1+\mu) X_{t-1} + \sqrt{\delta + \gamma X_{t-1}^2}\, z_t$

- $z \sim$ IID $N(0,1)$

Problem 1 $\boxed{16 \text{ points}}$

Consider $\mu = -1$, $\delta = 1$, $\gamma = \frac{1}{2}$, i.e.,

(1*) $X_t = \sqrt{1 + \frac{1}{2} X_{t-1}^2}\, z_t$

$$\left.\begin{aligned} &\text{Set} \quad \sigma_t^2 := 1 + \tfrac{1}{2} X_{t-1}^2, \\ &\text{then} \quad X_t = \sigma_t z_t \end{aligned}\right\} (2)$$

$\boxed{9}$ a) Claim: X is an ARCH(1)-process.

Proof:

We have to show:

(i) X stationary ($\frac{1}{2}$)

(ii) X has the representation ($\frac{1}{2}$)

$$X_t = \sigma_t z_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2, \quad \alpha_0 > 0, \alpha_1 \geq 0$$

(iii) $z_t \perp\!\!\!\perp (X_{t-j}, j \in \mathbb{N})$ — $\frac{1}{2}$
$z \sim IID\mathcal{N}(0,1)$

(iii) is satisfied by assumption.

To see (ii), set

$\alpha_0 := \delta = 1 > 0,$ $\frac{1}{2}$ $\frac{1}{2}$

$\alpha_1 := \sigma = \frac{1}{2} \geq 0,$

then (2) is the desired representation.

To show (i), we need to show

(a) $\mathbb{E}[X_t]$ is constant $\frac{1}{2}$

(b) $Var[X_t] < +\infty$ — $\frac{1}{2}$

(c) $Cov[X_{t+h}, X_{s+h}] = Cov[X_t, X_s]$ $\forall s, t, h \in \mathbb{Z}$ $\frac{1}{2}$

(a) $\mathbb{E}[X_t] = \mathbb{E}[\sqrt{1+\frac{1}{2}X_{t-1}^2}\, z_t]$

$\overset{z_t \perp\!\!\!\perp X_{t-1}}{=} \mathbb{E}[\sqrt{1+\frac{1}{2}X_{t-1}^2}]\, \mathbb{E}[z_t]$ $\frac{1}{2}$

$\overset{z_t \sim \mathcal{N}(0,1)}{=} \mathbb{E}[\sqrt{1+\frac{1}{2}X_{t-1}^2}] \cdot 0$ $\frac{1}{2}$

$\frac{1}{2}$ $= 0$ if $Var[X_t] = \mathbb{E}[X_t^2] < +\infty$, see (b)
by Hölder

(b) $Var[X_t] \overset{\mathbb{E}}{=} 2 < +\infty$ by assumption

(c) $Cov[X_{t+h}, X_{t+h}] = Var[X_{t+h}] = 2$ 1

$= Var[X_t] = Cov[X_t, X_t]$

Let now $s \neq t$, $t > s$ ($t < s$ follows by symmetry) $\frac{1}{2}$

$Cov[X_s, X_t] \overset{\mathbb{E}=0}{=} \mathbb{E}[X_s X_t]$

$\frac{1}{2}$ $= \mathbb{E}[X_s \sqrt{1+\frac{1}{2}X_{t-1}^2}\, z_t]$

$\frac{1}{2}$ $\overset{z_t \perp\!\!\!\perp X_s, X_{t-1}}{=} \mathbb{E}[X_s\sqrt{1+\frac{1}{2}X_{t-1}^2}]\, \mathbb{E}[z_t]$

$\overset{z_t \sim \mathcal{N}(0,1)}{=} \mathbb{E}[X_s\sqrt{1+\frac{1}{2}X_{t-1}^2}] \cdot 0$ $\frac{1}{2}$

$\frac{1}{2}$ $= 0$ since $Var[X_s] = 2 < +\infty$

$\Rightarrow \mathbb{E}[X_s \sqrt{1+\frac{1}{2}X_{t-1}^2}] \overset{\text{Hölder}}{\leq} \mathbb{E}[X_s^2]^{\frac{1}{2}}\, \mathbb{E}[1+\frac{1}{2}X_{t-1}^2]^{\frac{1}{2}}$

$= \sqrt{2} \cdot \sqrt{1+1} = 2$

Therefore we conclude that X is an ARCH(1)-process.

3 b) Claim: X is $WN(0,2)$.

Proof: We have to show that

- $\mathbb{E}[X_t]=0$ (1/2)
- $\gamma_X(h) = \begin{cases} 2 & \text{if } h=0 \\ 0 & \text{else} \end{cases}$ (1/2)

From a)(i)(a) we know that $\mathbb{E}[X_t]=0$ (1/2)

It follows from the definition of the ACVF that

$$\gamma_X(h) = Cov[X_t, X_{t+h}]$$ (1/2)

for arbitrary t, since X is stationary (1/2)

From a)(i)(c) we obtain

$$\gamma_X(h) = Cov[X_t, X_{t+h}] = \begin{cases} 2 & \text{if } h=0 \\ 0 & \text{else,} \end{cases}$$ (1/2)

which finishes the proof.

4 c) We show in Problem 2(c) that

$$\mathbb{E}[X_s^2X_t^2]-\mathbb{E}[X_s^2]\mathbb{E}[X_t^2]$$

$$\| $$

(1) $$Cov(X_s^2, X_t^2) = \mathbb{E}[(X_s^2-\mathbb{E}[X_s^2])(X_t^2-\mathbb{E}[X_t^2])]$$
$$= 2^{5-|t-s|} \quad \text{for } s \neq t$$

If X would be IID, then for all measurable f,g:

(1/2) $$\mathbb{E}[f(X_s)g(X_t)] = \mathbb{E}[f(X_s)]\mathbb{E}[g(X_t)] \quad \text{for } s\neq t$$

(1/2) Set $f(x) := g(x) := x^2$

(1/2) $\Rightarrow$ $\mathbb{E}[X_s^2X_t^2] = \mathbb{E}[X_s^2]\mathbb{E}[X_t^2]$

(1/2) $\Rightarrow$ $Cov[X_s^2, X_t^2] = 0$,

(1/2) which is a contradiction to Prob. 2(c)

(1/2) $\Rightarrow$ X is <u>not</u> IID.

Problem 2 $\boxed{44 \text{ points}}$

$\boxed{1}$ a) $\mathbb{E}[X_t^2] = \text{Var}[X_t] = 2$ (1)

$\boxed{3{,}5}$ b) $\mathbb{E}[X_t^4] = \mathbb{E}[((1+\frac{1}{2}X_{t-1}^2)Z_t^2)^2]$

$$= \mathbb{E}[(1+\tfrac{1}{2}X_{t-1}^2)^2 Z_t^4] \quad (\tfrac{1}{2})$$

$$\overset{Z_t \perp\!\!\!\perp X_s,\ s<t}{=} \mathbb{E}[(1+\tfrac{1}{2}X_{t-1}^2)^2]\ \mathbb{E}[Z_t^4] \quad (\tfrac{1}{2})$$

$$\overset{Z_t \sim N(0,1)}{=} \mathbb{E}[1 + X_{t-1}^2 + \tfrac{1}{4}X_{t-1}^4] \cdot 3 \quad (\tfrac{1}{2})$$

$$\overset{\mathbb{E}\ \text{lin}}{=} 3 + 3\,\mathbb{E}[X_{t-1}^2] + \tfrac{3}{4}\,\mathbb{E}[X_{t-1}^4]$$

$$\overset{a)}{=} 3 + 6 + \tfrac{3}{4}\,\mathbb{E}[X_{t-1}^4] \quad (\tfrac{1}{2})$$

$$\overset{\mathbb{E}[X_t^4]\ \text{const}}{=} 9 + \tfrac{3}{4}\,\mathbb{E}[X_t^4] \quad (\tfrac{1}{2})$$

$$\Leftrightarrow (1-\tfrac{3}{4})\,\mathbb{E}[X_t^4] = 9 \quad (\tfrac{1}{2})$$

$$\Leftrightarrow \boxed{\mathbb{E}[X_t^4] = 9 \cdot 4 = 36} \quad (\tfrac{1}{2})$$

$\boxed{4{,}5}$ c) Let t be fixed and $h > 0$

$$\text{Cov}[X_t^2, X_{t+h}^2] = \mathbb{E}[X_t^2 X_{t+h}^2] - \mathbb{E}[X_t^2]\,\mathbb{E}[X_{t+h}^2]$$

$$\overset{a)}{=} \mathbb{E}[X_t^2 X_{t+h}^2] - 4 \quad (\tfrac{1}{2})$$

$$= \mathbb{E}[X_t^2(1+\tfrac{1}{2}X_{t+h-1}^2)Z_{t+h}^2] - 4 \quad (\tfrac{1}{2})$$

$$\overset{Z_t \perp\!\!\!\perp X_s,\ s<t}{=} \mathbb{E}[X_t^2 + \tfrac{1}{2}X_t^2 X_{t+h-1}^2]\ \mathbb{E}[Z_{t+h}^2] - 4 \quad (\tfrac{1}{2})$$

$$\overset{Z \sim N(0,1),\ a),\ \text{lin}\ \mathbb{E}}{=} (2 + \tfrac{1}{2}\,\mathbb{E}[X_t^2 X_{t+h-1}^2]) \cdot 1 - 4 \quad (\tfrac{1}{2})$$

$$= 2 + \tfrac{1}{2}(\text{Cov}[X_t^2, X_{t+h-1}^2] + 4) - 4 \quad (\tfrac{1}{2})$$

$$= \tfrac{1}{2}\,\text{Cov}[X_t^2, X_{t+h-1}^2] \quad (\tfrac{1}{2})$$

$$\overset{\text{rec}}{=} \dots\ 2^{-h}\,\text{Cov}[X_t^2, X_t^2] \quad (1)$$

$$= 2^{-h} \cdot \left(\mathbb{E}[X_t^4] - \mathbb{E}[X_t^2]^2 \right) \quad \tfrac{1}{2}$$

$$\overset{b),a)}{=} 2^{-h} \left(36 - 4 \right)$$

$$= 2^{-h} \; 2^5$$

$$= \underline{\underline{2^{5-h}}} \quad \tfrac{1}{2}$$

Let now $h < 0$, then

$$\mathrm{Cov}[X_t^2, X_{t+h}^2] = \mathrm{Cov}[X_t^2, X_{t-(-h)}^2]$$

Set $\tilde{t} := t-(-h)$

$$= \mathrm{Cov}[X_{\tilde{t}+(-h)}^2, X_{\tilde{t}}^2] \quad 1$$

$$\overset{\text{sym. cov.}}{=} \mathrm{Cov}[X_{\tilde{t}}^2, X_{\tilde{t}+\underbrace{(-h)}_{>0}}^2]$$

$$\overset{-h>0}{=} 2^{5-(-h)}$$

For $h=0$

$$\mathrm{Cov}[X_t^2, X_t^2] = \mathrm{Var}[X_t^2]$$

$$= \mathbb{E}[X_t^4] - \mathbb{E}[X_t^2]^2 \quad 1$$

$$\overset{b),a)}{=} 32 = 2^5$$

$\Rightarrow$ We conclude

$$\boxed{\mathrm{Cov}[X_t, X_{t+h}] = 2^{5-|h|}}, \quad \tfrac{1}{2}$$

which does not depend on t.

$\boxed{5{,}5}$ e) Set $\tilde{Z}_t = X_t^2 - \sigma_t^2$

$$= X_t^2 - \left(1 + \tfrac{1}{2} X_{t-1}^2\right)$$

Then

$$\mathbb{E}[\tilde{Z}_t] = \mathbb{E}[X_t^2 - \sigma_t^2]$$

$$\overset{\text{lin.}}{=} \mathbb{E}[X_t^2] - 1 - \tfrac{1}{2}\mathbb{E}[X_{t-1}^2] \quad \tfrac{1}{2}$$

$$\overset{a)}{=} 2 - 1 - \tfrac{1}{2} \cdot 2$$

$$= 0 \quad \tfrac{1}{2}$$

$$\mathrm{Cov}[\tilde{z}_s, \tilde{z}_t] \overset{\mathbb{E}=0}{=} \mathbb{E}[\tilde{z}_s \tilde{z}_t] \quad (1/2)$$

$$= \mathbb{E}\left[(X_s^2 - 1 - \tfrac{1}{2} X_{s-1}^2)(X_t^2 - 1 - \tfrac{1}{2} X_{t-1}^2) \right] \quad (1/2)$$

$$= \mathbb{E}[X_s^2 X_t^2 - X_s^2 - \tfrac{1}{2} X_s^2 X_{t-1}^2]$$
$$- \mathbb{E}[X_t^2 - 1 - \tfrac{1}{2} X_{t-1}^2]$$
$$- \tfrac{1}{2} \mathbb{E}[X_{s-1}^2 X_t^2 - X_{s-1}^2 - \tfrac{1}{2} X_{s-1}^2 X_{t-1}^2] \quad (1/2)$$

$$\overset{\mathbb{E}[\tilde{z}_t]=0}{=} \mathrm{Cov}[X_s^2, X_t^2] + \cancel{4} - \cancel{2} - \tfrac{1}{2} \mathrm{Cov}[X_s^2, X_{t-1}^2] - \cancel{\tfrac{1}{2} 4}$$
$$- 0$$
$$- \tfrac{1}{2} \left(\mathrm{Cov}[X_{s-1}^2, X_t^2] + \cancel{4} - \cancel{2} - \tfrac{1}{2} \mathrm{Cov}[X_{s-1}^2, X_{t-1}^2] - \cancel{\tfrac{1}{2} 4} \right) \quad (1/2)$$

$$\overset{c)}{=} (1 + \tfrac{1}{4}) 2^{5-|t-s|} - \tfrac{1}{2} 2^{5-|t-s-1|} - \tfrac{1}{2} 2^{5-|t+1-s|} \quad (1/2)$$

$$= 5 \cdot 2^{5-2-|t-s|} - 2^{4-|t-s-1|} - 2^{4-|t+1-s|} \quad (1/2)$$

$$= \begin{cases} 5 \cdot 2^3 - 2^3 - 2^3 = 24 & t=s \\ 2^{3-|t-s|}(5 - 2^{1+1} - 2^{1-1}) = 0 & t>s \\ 2^{3-|t-s|}(5 - 2^{1-1} - 2^{1+1}) = 0 & t<s \end{cases} \quad (1)$$

$\Rightarrow \tilde{z}$ is $WN(0, 24)$ (1/2)

1,5 d) From a) $\mathbb{E}[X_t^2] = 2$ is constant (1/2)

From b) $\mathrm{Var}[X_t^2] = \mathbb{E}[X_t^4] - \mathbb{E}[X_t^2]^2$

$$= 36 - 4$$
$$= 32 < +\infty \quad (1/2)$$

From c) $\mathrm{Cov}[X_t^2, X_s^2] = 2^{5-|t-s|}$

$$= 2^{5-|t+h-(s+h)|}$$
$$= \mathrm{Cov}[X_{t+h}, X_{s+h}] \quad (1/2)$$

Therefore all conditions for a stationary TS are satisfied

4 f) We have to show that we find a representation

$$(\tilde{X}_t - \mathbb{E}[\tilde{X}_t]) - \phi_1 (\tilde{X}_{t-1} - \mathbb{E}[\tilde{X}_{t-1}]) = \tilde{Z}_t, \quad \tfrac{1}{2}$$

$\tilde{Z} \sim WN(0, \sigma^2)$, since we have already shown that X^2 is stationary. $\tfrac{1}{2}$

We have:

$$\sigma_t^2 = X_t^2 - \tilde{Z}_t \qquad \tfrac{1}{2}$$
$$= 1 + \tfrac{1}{2} X_{t-1}^2 \qquad \text{by construction}$$

$$\Rightarrow \quad X_t^2 - \tilde{Z}_t \overset{\frac{1}{2}}{=} 1 + \tfrac{1}{2} X_{t-1}^2 \overset{\frac{1}{2}}{=} 2 + \tfrac{1}{2}(X_{t-1}^2 - 2)$$

$$\overset{\frac{1}{2}}{\Leftrightarrow} \quad (X_t^2 - 2) - \tfrac{1}{2}(X_{t-1}^2 - 2) = \tilde{Z}_t,$$

where $\tilde{Z} \sim WN(0, 24)$ by e). $\tfrac{1}{2}$

Set $\phi_1 := \frac{1}{2}$ and we are done. $\tfrac{1}{2}$

5.5 g) Given X_1^2 and X_2^2, we derive the best linear predictor $b_3^\ell((X_1^2, X_2^2))$ for X_3^2 from Proposition 2.3.5.

$$b_3^\ell((X_1^2, X_2^2)) = a_0 + a_1 X_2^2 + a_2 X_1^2 \qquad \tfrac{1}{2}$$

$$a_0 = \mathbb{E}[X_t^2](1 - a_1 - a_2) = 2(1 - a_1 - a_2) \qquad \tfrac{1}{2}$$

$$\begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \gamma(1) \\ \gamma(2) \end{pmatrix} \qquad \tfrac{1}{2}$$

$$\Leftrightarrow \quad 2^5 \cdot \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 2^5 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} \qquad \tfrac{1}{2}$$

$$\Leftrightarrow \left(\begin{array}{l} a_1 + \frac{1}{2} a_2 = \frac{1}{2} \\ \frac{1}{2} a_1 + a_2 = \frac{1}{4} \end{array} \right) \qquad \tfrac{1}{2}$$

$$\Rightarrow \quad \left(\tfrac{1}{2} - 2\right) a_2 = \tfrac{1}{2} - 2\,\tfrac{1}{4} = 0$$

$$\Leftrightarrow \quad -\tfrac{3}{2}\, a_2 = 0$$

$$\Leftrightarrow \quad \boxed{a_2 = 0} \quad \tfrac{1}{2}$$

$\tfrac{1}{2}$

$$\Rightarrow \quad a_1 + \tfrac{1}{2} \cdot 0 = \tfrac{1}{2}$$

$$\Leftrightarrow \quad \boxed{a_1 = \tfrac{1}{2}} \quad \tfrac{1}{2}$$

$$\Rightarrow \quad \boxed{b_3^\ell\left((X_1^2, X_2^2)\right) = 1 + \tfrac{1}{2} X_2^2} \quad \tfrac{1}{2}$$

since

$$\boxed{a_0 = 2\left(1 - \tfrac{1}{2}\right) = 1} \quad \tfrac{1}{2}$$

2

b) $$MSE\left[b_3^\ell\left((X_1^2, X_2^2)\right), X_3^2\right] \overset{\frac{1}{2}}{=} \mathbb{E}\left[\left(b_3^\ell\left((X_1^2, X_2^2)\right) - X_3^2\right)^2\right]$$

$$\overset{\frac{1}{2}}{=} \mathbb{E}\left[\left(1 + \tfrac{1}{2} X_2^2 - \left(2 + \tfrac{1}{2}(X_2^2 - 2) + \tilde{Z}_3\right)\right)^2\right]$$

$$\tfrac{1}{2} \quad = \mathbb{E}\left[\tilde{Z}_3^2\right]$$

$$\tfrac{1}{2} \quad \overset{c)}{=} 24$$

5

i) Let X_1, X_2 be given.

Compute

$$X_3^{(1)} = \sqrt{1 + \tfrac{1}{2} X_2^2}\; Z_3^{(1)} \quad 1$$

$$X_3^{(2)} = \sqrt{1 + \tfrac{1}{2} X_2^2}\; Z_3^{(2)}$$

(*) Set

$$X_2(1) := \tfrac{1}{2}\sqrt{1 + \tfrac{1}{2} X_2^2}\left(Z_3^{(1)} + Z_3^{(2)}\right) \quad 1$$

and

$$\left(X_2(1)\right)^2 = \tfrac{1}{4}\left(1 + \tfrac{1}{2} X_2^2\right)\left(Z_3^{(1)} + Z_3^{(2)}\right)^2 \quad 1$$

or

(**)

$$X_2^2(1) = \frac{1}{2}\left(X_3^{(1)2} + X_3^{(2)2}\right) \quad 1$$

$$= \frac{1}{2}\left(1+\frac{1}{2}X_2^2\right)\left(z_3^{(1)2} + z_3^{(2)2}\right) \quad 1$$

$\boxed{5.5}$ j) $\mathrm{MSE}[(X_2(1))^2, X_3^2] \overset{1/2}{=} \mathbb{E}\left[\left(\frac{1}{4}\left(1+\frac{1}{2}X_2^2\right)\left(z_3^{(1)}+z_3^{(2)}\right)^2 - \left(1+\frac{1}{2}X_2^2\right)z_3^2\right)^2\right]$

$$1/2 \qquad = \mathbb{E}\left[\left(1+\frac{1}{2}X_2^2\right)^2\left(\frac{1}{4}\left(z_3^{(1)}+z_3^{(2)}\right)^2 - z_3^2\right)^2\right]$$

$$1/2 \qquad \overset{X_2 \perp z_3 \perp z_3^{(i)}}{=} \mathbb{E}\left[\left(1+\frac{1}{2}X_2^2\right)^2\right]\mathbb{E}\left[2^{-4}\left(z_3^{(1)}+z_3^{(2)}\right)^4 - \frac{1}{2}\left(z_3^{(1)}+z_3^{(2)}\right)^2 z_3^2 + z_3^4\right]$$

$z_3^{(1)}+z_3^{(2)} \sim N(0,2)$

$3\cdot 2^2$

$$\overset{(*)}{=} 12\cdot\left(2^{-4}\cdot 12 - \frac{1}{2}\mathbb{E}\left[\left(z_3^{(1)}+z_3^{(2)}\right)^2\right]\mathbb{E}[z_3^2] + 3\right)$$

$$1/2 \qquad = 12\cdot\left(\frac{3}{4} - \frac{1}{2}\cdot 2\cdot 1 + 3\right)$$

$$= \cancel{12}^{3}\cdot\frac{11}{\cancel{4}} = 33 \quad 1/2$$

$$\mathrm{MSE}[X_2^2(1), X_3^2] \overset{1/2}{=} \mathbb{E}\left[\left(\frac{1}{2}\left(1+\frac{1}{2}X_2^2\right)\left(z_3^{(1)2}+z_3^{(2)2}\right) - \left(1+\frac{1}{2}X_2^2\right)z_3^2\right)^2\right]$$

$$1/2 \qquad \overset{X_2 \perp z_3^{(i)}}{=} \mathbb{E}\left[\left(1+\frac{1}{2}X_2^2\right)^2\right]\mathbb{E}\left[\frac{1}{4}\left(z_3^{(1)2}+z_3^{(2)2} - 2z_3^2\right)^2\right]$$

$$\overset{(*)}{=} 12\cdot\left(\frac{1}{4}\mathbb{E}\left[\left(z_3^{(1)2}+z_3^{(2)2}\right)^2 - 4\left(z_3^{(1)2}+z_3^{(2)2}\right)z_3^2 + 4z_3^4\right]\right)$$

properties of normal distribution

$$1/2 \qquad = 12\cdot\left(\frac{1}{4}\mathbb{E}\left[z_3^{(1)4} + 2z_3^{(1)2}z_3^{(2)2} + z_3^{(2)4}\right] - (1\cdot 1 + 1\cdot 1) + 3\right)$$

$$= 12\cdot(2+1)$$

$$1/2 \qquad = 36$$

$$(*)\ \mathbb{E}\left[\left(1+\frac{1}{2}X_2^2\right)^2\right] = 1 + \mathbb{E}[X_2^2] + \frac{1}{4}\mathbb{E}[X_2^4]$$

$$\overset{\text{Prob 2b}}{=} 1 + 2 + \frac{1}{4}\cdot 36 = 12 \quad 1$$

3k) The mean squared errors for the three are

$$b_3^e((X_1^2, X_2^2)) \quad : \quad 24$$
$$X_2(1)^2 \quad : \quad 33 \qquad \text{e.g. } (1)$$
$$X_2^2(1) \quad : \quad 36$$

The best linear predictor has the smallest MSE
$X_2(1)^2$ performs better than $X_2^2(1)$. e.g. (1)
This means that it is better to estimate $X_2(1)$, the predictor of X_3 and then square it instead of computing the sample mean of the squared predictor samples. e.g. (1)

At the same time it holds that

- $$\mathbb{E}[X_3^2] = \mathbb{E}[(1+\tfrac{1}{2}X_2^2)Z_3^2] \overset{\perp\!\!\!\perp}{=} \mathbb{E}[(1+\tfrac{1}{2}X_2^2)]\underbrace{\mathbb{E}[Z_3^2]}_{=1}$$
$$= \mathbb{E}[(1+\tfrac{1}{2}X_2^2)]$$
$$= \mathbb{E}[b_3^e((X_1^2, X_2^2))]$$

- $$\mathbb{E}[X_2^2(1)] = \mathbb{E}[\tfrac{1}{2}(1+\tfrac{1}{2}X_2^2)(Z_3^{(1)2} + Z_3^{(2)2})]$$
$$\overset{\perp\!\!\!\perp}{=} \frac{1}{2}\mathbb{E}[(1+\tfrac{1}{2}X_2^2)]\cdot(\underbrace{\mathbb{E}[Z_3^{(1)2}]}_{=1} + \underbrace{\mathbb{E}[Z_3^{(2)2}]}_{=1})$$
$$= \mathbb{E}[X_3^2]$$

BUT

- $$\mathbb{E}[X_2(1)^2] = \mathbb{E}[\tfrac{1}{4}(1+\tfrac{1}{2}X_2^2)(Z_3^{(1)} + Z_3^{(2)})^2]$$
$$\overset{\perp\!\!\!\perp}{=} \frac{1}{4}\mathbb{E}[(1+\tfrac{1}{2}X_2^2)]\ \mathbb{E}[Z_3^{(1)2} + 2Z_3^{(1)}Z_3^{(2)} + Z_3^{(2)2}]$$
$$\overset{\text{lin}, \perp\!\!\!\perp}{=} \frac{1}{4}\mathbb{E}[X_3^2]\cdot(1 + 2\underbrace{\mathbb{E}[Z_3^{(1)}]}_{=0}\underbrace{\mathbb{E}[Z_3^{(1)}]}_{=0} + 1)$$
$$= \frac{1}{2}\mathbb{E}[X_3^2],$$

i.e, $(X_2(1))^2$ is biased.

So $X_2(1)^2$ has a smaller MSE but at the same time underestimates on average. The experimenter must decide what to prefer.