Extra Exercises in Basic Probability for Financial Time Series

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Easier exercises

Ex. 1 — For $t \in \mathbb{Z}$, let $X_t = Z_t + 0.5Z_{t-1}$ where Z_t is an iid sequence with mean 0 and variance σ^2.

a) Compute the variance of the sample mean $(X_1 + X_2 + X_3 + X_4)/4$.

b) Let $Y_t = Z_t - 0.5Z_{t-1}$. Compute the variance of the sample mean $(Y_1 + Y_2 + Y_3 + Y_4)/4$ and compare to the previous part of the exercise.

Ex. 2 — For $t \in \mathbb{Z}$, let $X_t = X_0$ for all $t \in \mathbb{Z}$ where X_0 is a random variable with finite variance and zero mean. Is the process $(X_t, t \in \mathbb{Z})$

a) strictly stationary?

b) weakly stationary?

Ex. 3 — Let $(X_t, t \in \mathbb{Z})$ be a sequence of random variables with (not necessarily identical) finite variance. For $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, show that

$$\mathsf{Var}(\sum_{i=1}^{n} \alpha_i X_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \, \mathsf{Cov}(X_i, X_j).$$

You may assume for simplicity that X_t all have zero mean.

Ex. 4 — Let $(X_t, t \in \mathbb{N})$ be a sequence of zero-mean iid random variables with unit variance. Let $Y_1 = X_1$ and let $Y_t = Y_{t-1} + X_t$ for $t \in \mathbb{N}$ with $t > 1$. Compute $\gamma_Y(1, 2)$ and $\gamma_Y(2, 3)$. Is the process $(Y_t, t \in \mathbb{N})$ stationary?

Ex. 5 — Let $U \sim \mathcal{U}([0, 1])$.

a) Find the density function of $S = U^2$.

b) Find the density function of $T = -\log(U)/\lambda$, where $\lambda > 0$.

Ex. 6 — Let $(X_t, t \in \mathbb{Z})$ be a stationary process with autocovariance function γ. Use the Cauchy-Schwarz inequality to show that for all $h \in \mathbb{Z}$, $|\gamma(h)| \leq |\gamma(0)|$. Assume for simplicity that $(X_t, t \in \mathbb{Z})$ has zero mean.

Ex. 7 — Let $X = U_1 + U_2 + \ldots + U_{12} - 6$ where, for $i = 1, \ldots, 12$, $U_i \sim \mathcal{U}([0, 1])$ are mutually independent of one another.

a) Calculate the mean and variance of X.

b) Use the CLT to explain how X can be used to generate approximate standard normal random variables.

Harder exercises

Ex. 8 — Assume that the random variables X and Y are continuous with joint density $f_{X,Y}(x,y)$ and finite means.

a) Show that $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.

b) Show that $X \geq 0 \implies \mathbb{E}[X|Y] \geq 0$.

c) Assume that X and Y are independent and show that $\mathbb{E}[X|Y] = \mathbb{E}[X]$.

Ex. 9 — Let B have the Bernoulli distribution with parameter $p = 1/2$ and let $X \sim \mathcal{N}(0,1)$ be independent of B. Let $Y = (2B-1)X$.

a) Compute $\mathbb{E}[Y]$.

b) Compute for $P(Y \leq y)$ for $y \in \mathbb{R}$. What is the distribution of Y? *Hint: Note that* $P(Y \leq y) = P(Y \leq y, B = 0) + P(Y \leq y, B = 1)$.

c) Compute $\mathsf{Cov}(X,Y)$. Are X and Y independent?

Ex. 10 — Let $X = \exp(Z)$ where $Z \sim \mathcal{N}(0,1)$ (this is called a lognormal random variable).

a) Find the density function of X

b) Use this density to find the mean and variance of X.

Ex. 11 — Let $\mathcal{X}$ be the family of all (real-valued) random variables with finite variance on the probability space $(\Omega, \mathcal{A}, P)$. Show that $\mathcal{X}$ is a vector space. *Hint: Use the* Cauchy-Schwarz inequality $|\,\mathbb{E}[XY]|^2 \leq \mathbb{E}[X^2]\,\mathbb{E}[Y^2]$. *You may assume that if X and Y are random variables (i.e. measurable), then so are $\alpha X + \beta Y$ for $\alpha, \beta \in \mathbb{R}$*