Answers to selected exercises

"A First Course in Stochastic Models", Henk C. Tijms

1.1 $P(N(2) \geq 1) = P(\gamma_t \leq 2) = 1 - e^{-2\lambda}$

1.2 (a) Let W_j = waiting time if j passengers already arrived, $j = 0,1,\ldots,6$. Then

$P(W_j \leq x) = 1 - \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x}$, $k = 6 - j, j = 0,1,\ldots,6$

(b) $P(\text{no wait}) = \begin{cases} 1 & \text{if } n \, mod \, 7 = 1 \\ 0 & \text{if } n \, mod \, 7 \neq 1 \end{cases}$

(c) Long-run fraction for $j = 0,1,\ldots,6$ is $1/7$

(d) Let W = waiting time. Then

$$P(W > x) = \frac{1}{7} \sum_{j=0}^{\infty} \sum_{k=0}^{5-j} e^{-\lambda x} \frac{(\lambda x)^k}{k!}$$

1.3 (a) Let W_j = waiting time if j passengers already arrived, $j = 0,1,\ldots,6$.

Let $N(x)$ be a $Poi(\lambda)$-process where a passenger arrives every second "event"

$$P(W_j > x) = P(N(x) \leq 2(5-j)+1) = \sum_{k=0}^{2(5-j)+1} e^{-\lambda x} \frac{(\lambda x)^k}{k!}$$

(b) Same as 1.2b.

(c) Same as 1.2c.

1.4 $P(\text{take bus 1}) = \int_0^{10} P(\text{take bus 1}|\text{arrive at } x) P(\text{arrive at } x) dx =$

$$= \int_0^{10} e^{-\frac{1}{10}(10-x)} \frac{1}{10} dx = 1 - \frac{1}{e} \approx 0.63$$

Why > 0.5? Since $E[X_1] = 5$ and $E[X_3] = 10$.

1.5 Let X =number of cars passed before you can cross

$P(X = n) = \left(1 - e^{-\lambda c}\right)^n e^{-\lambda c}$

Thus, the geometric distribution with $p = e^{-\lambda c}$.

1.6 δ_t = time since the last arrival before or at t

(a)

$$P(\delta_t \le x) = \sum_{n=1}^{\infty} P(t - x \le \delta_n \le t, t < S_{n+1}) = 1 - e^{-\lambda x}$$

(b)

$$P(\delta_t = t, \gamma_t \le u) = e^{-\lambda t}\left(1 - e^{-\lambda u}\right) = P(\delta_t = t)P(\gamma_t \le u)$$

and for $0 \le v < t$,

$$P(\delta_t \le v, \gamma_t \le u) = \left(1 - e^{-\lambda v}\right)\left(1 - e^{-\lambda u}\right) = P(\delta_t \le v)P(\gamma_t \le u)$$

1.7 Let $T =$ travel time of a given fast car.

$$E[T] = E[T|\text{slow car leaves before caught up}]P(\text{leaves before}) + E[T|\text{fast car catches up}]P(\text{catches up}) =$$

$$= \frac{L}{s_1} e^{-\lambda_2\left(\frac{L}{s_2} - \frac{L}{s_1}\right)} + \int_0^{\frac{L}{s_2} - \frac{L}{s_1}} \left(\frac{L}{s_2} - u\right)\left(\lambda_2 e^{-\lambda_2 u}\right) du$$

$$= \frac{L}{s_2} - \frac{1}{\lambda_2}\left(1 - e^{-\lambda_2\left(\frac{L}{s_2} - \frac{L}{s_1}\right)}\right)$$

1.8 Condition on $X_1 + \cdots + X_n$ and $X_{n+2} + \cdots X_{n+k+1}$ and compute

$$P(X_1 + \cdots X_n < t < X_1 + \cdots X_{n+1}, X_1 + \cdots X_{n+k+1} > t + s)$$
$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{j=0}^{k} e^{-\lambda s} \frac{(\lambda s)^j}{j!} = P(N(t) = n)P(N(s) \le k)$$

1.9 (a) The merged process $N(t) = N_1(t) + N_2(t)$ is also Poisson, $\mathrm{Poi}(\lambda_1 + \lambda_2)$ and

$$P(N(1) = n) = e^{-(\lambda_1 + \lambda_2)} \frac{(\lambda_1 + \lambda_2)^n}{n!}$$

(b) Let $X =$ service time of the next request.

$$f_X(x) = \frac{\lambda_1}{\lambda_1 + \lambda_2} \mu_1 e^{-\mu_1 x} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \mu_2 e^{-\mu_2 x}$$

1.10 Let $N(t) =$ number of claims.

$$P(N(1) = k) = \int_0^{\infty} P(N(1) = k|\lambda = x) f_\lambda(x) dx =$$

$$= \int_0^{\infty} e^{-x} \frac{x^k}{k!} \frac{\beta^\alpha x^{\alpha - 1}}{\Gamma(\alpha)} e^{-\beta x} dx$$

$$= \frac{\Gamma(\alpha + k)}{\Gamma(k + 1)\Gamma(\alpha)} \int_0^{\infty} \frac{1}{\Gamma(\alpha + k)} e^{-x(\beta + 1)} \beta^\alpha x^{k + \alpha - 1} dx$$

[Let $\delta = k + \alpha$ and $\gamma = \beta + 1$]

$$= \frac{\Gamma(\alpha+k)}{\Gamma(k+1)\Gamma(\alpha)} \frac{\beta^\alpha}{\gamma^\delta} \int_0^\infty \frac{1}{\Gamma(\delta)} e^{-\gamma x} x^{\delta-1} \gamma^\delta dx$$

$$= \frac{\Gamma(\alpha+k)}{\Gamma(k+1)\Gamma(\alpha)} \left(\frac{\beta}{\beta+1}\right)^\alpha \left(\frac{1}{\beta+1}\right)^k$$

2.1 If $N(t) =$ the number of lamps replaced at time t, then the expected number of street lamps used in $[0, T]$ is $M(T) + 1$ where

$$M(T) = E[N(T)] = \sum_{n=1}^{\infty} \left(1 - \sum_{k=0}^{2n-1} e^{-\lambda T} \frac{(\lambda T)^k}{k!}\right)$$

2.2 (a) Let $N(t) =$ number of weeks up to waste amount t. Then the expected number of weeks needed to exceed L is $M(L) + 1$ where

$$M(L) = \sum_{n=1}^{\infty} P(S_n \leq L) = \sum_{n=1}^{\infty} \frac{1}{\Gamma(n\alpha)} \int_0^{\lambda L} e^{-u} u^{\alpha-1} du$$

(b)

$$M(L) \approx \frac{\lambda L}{\alpha} + \frac{1}{2} \frac{\alpha+1}{\alpha} - 1$$

2.3 Mean and standard deviation are given by 7.78 and 4.78 minutes, respectively.

2.4 Use the inequalities

$$\begin{aligned} P(X_1 + \cdots X_n \leq t) &\leq P(X_1 + \cdots + X_k \leq t) P(X_{k+1} + \cdots X_n \leq t) \\ &\leq P(X_1 + \cdots + X_k \leq t) P(X_{k+1} \leq t) \cdots P(X_n \leq t) \end{aligned}$$

to conclude that $F_n(t) \leq F_\alpha(t) \big(F(t)\big)^{n-k}$ for $n > k$ and $t \geq 0$. (b) follows immediately from this.

2.5 (a) If $N_p(t)$ denotes a $Poi(\lambda)$-process, then

$$P(N(t) > k) = P(N(t) \geq k+1) = P(N_p(t) \geq (k+1)r)$$

(b) The excess variable is $\gamma_t = \delta_{N(t)+1} - 1$. Let $\delta_{N(t)} = 0$ (using memoryless property).

$$P(N(t) \leq k) = P(N_p(t) \leq kr)$$

Thus γ_t is Erlang if and only if $kr - j$ $Poi(\lambda)$-arrivals occurred in $(0, t)$, giving the formula.

2.6 (a) The long-run fraction of time the process is in state i equals $(p_i/\lambda_i)(p_1/\lambda_1 + p_2/\lambda_2)$. This is also the limit of $\lim_{t\to\infty} P(X(t) = i)$.

3.1 Let X_n = system state at the beginning of nth day. State space

0 = both parts failed
1 = one part is working
2 = both parts are working

Transition matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0.50 & 0.50 & 0 \\ 0.25 & 0.25 & 0.50 \end{pmatrix}$$

3.2 Let X_n = system state at day n with state space

$(0,0)$ = both machines work
$(0,1)$ = one works, one is in day one of repair
$(0,2)$ = one works, one is in day two of repair
$(1,2)$ = one is in day one of repair, one is in day two of repair

Transition matrix

$$\begin{pmatrix} 9/10 & 1/10 & 0 & 0 \\ 0 & 0 & 9/10 & 1/10 \\ 9/10 & 1/10 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3.3 (a) Let X_n = number of containers at day n. State space $I = \{0,1,2,\dots\}$. Let

$$a = P(\text{container still present in eve}|\text{present in morning}) = P(\text{residency} > 1 \text{ day}) = e^{-\mu}$$

Transition probabilities $p_{ij} = \binom{i+1}{j} a^j (1-a)^{i+1-j}$ for $j = 0,1,\dots,i+1$ and 0 otherwise.

(b) Let a container be type-k if its residency time is $Exp(\mu_k)$, $k = 1,2$.

Let X_{nk} = number of type-k containers day n

$\{X_n = (X_{n1}, X_{n2})\}$ is a Markov chain with state space $I = \{(i_1, i_2): i_1, i_2 = 0,1,2,\dots\}$

Let $a_k = P(\text{type-}k\ \text{ present at eve}|\text{present in morning}) = e^{-\mu k}$

Then the transition probabilities are

$$p_{(i_1,i_2),(j_1,j_2)} = p\binom{i_1+1}{j_1} a_1^{j_1}(1-a_1)^{i_1+1-j_1} + (1-p)\binom{i_2+1}{j_2} a_2^{j_2}(1-a_2)^{i_2+1-j_2}$$

for $j_1 = 0,1,\dots,i_1+1$ and $j_2 = 0,1,\dots,i_2+1$, and 0 otherwise.

3.4 Define a Markov chain with an absorbing state.

0 = no game played yet
1 = one team won current game but not previous
2 = one team won last two but not the one before that

3 =one team won last three games

Let 3 be an absorbing state. Transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then $P(\text{exactly } m \text{ games}) = p_{03}^{(m)} - p_{03}^{(m-1)}$ where $p_{03}^{(m)} = P(X_m = 3|X_0 = 0)$.

3.5 (a) Let $p =$ probability that team A wins. Let the states be for $i = 1,2,3$

$(0,0) =$ no game played yet
$(i, A) =$ A has won the last i games but not the one before that
$(i, B) =$ B has won the last i games but not the one before that

Let $(3, A)$ and $(3, B)$ be absorbing states. We get a 7×7 transition matrix with

$p_{(0,0),(1,A)} = p \quad p_{(0,0),(1,B)} = 1 - p$
$p_{(i,A),(i+1,A)} = p \quad p_{(i,B),(i+1,B)} = 1 - p$
$p_{(i,A),(1,B)} = 1 - p \quad p_{(i,B),(1,A)} = p$
$p_{(3,A),(3,A)} = 1 \quad p_{(3,B),(3,B)} = 1$

All other transition probabilities are 0.

$$P(\text{more than } m \text{ games needed}) = 1 - (p_{(0,0),(3,A)}^{(m)} - p_{(0,0),(3,B)}^{(m)})$$

Let $f(i,j) = P(\text{A ultimate winner}|\text{current state is } (i,j))$, $i = 0,1,2,3$, $j = A, B$.Then

$$P(\text{A ultimate winner}) = f(0,0)$$

which is given by solving the linear equation system

$f(0,0) = pf(1, A) + (1 - p)f(1, B)$
$f(1, A) = pf(2, A) + (1 - p)f(1, B)$
$f(1, B) = pf(1, A) + (1 - p)f(2, B)$
$f(2, A) = pf(3, A) + (1 - p)f(1, B)$
$f(2, B) = pf(1, A) + (1 - p)f(3, B)$
$f(3, A) = 1$
$f(3, B) = 0$

$$f(0,0) = \frac{p^3(3 - 3p + p^2)}{1 - 2p + p^2 + 2p^3 - p^4}$$

(b) Let $d =$ probability of a draw

Then in the transition probabilities we add

$$p_{(0,0),(0,0)} = p_{(i,A),(0,0)} = p_{(i,B),(0,0)} = d$$

and $1-p$ is replaced by $1-p-d$ everywhere. Similarly in the linear equations, and the term $df(0,0)$ is added to the right-hand side of every equation.

3.6 Define states as

$0 =$ start or last toss was tails
$i =$ last i tosses were heads but not the one before, $i = 1,2,3$

Let $i = 3$ be an absorbing state. The transition probabilities become

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $\mu_i =$ expected number of throws to reach state 3 from state $i = 1,2$. Which gives an equation system

$\mu_0 = 1 + \frac{1}{2}\mu_0 + \frac{1}{2}\mu_1$

$\mu_1 = 1 + \frac{1}{2}\mu_0 + \frac{1}{2}\mu_2$

$\mu_2 = 1 + \frac{1}{2}\mu_0 + \frac{1}{2}\mu_3$

$\mu_3 = 0$

Solving this gives $\mu_0 = 14$. Since the gain is 12 the game is not fair.

3.7 Define states $i = 0,1,2,\dots,6$ where state i means that i of the six outcomes have appeared so far. State 6 is absorbing. The transition probabilities become

$p_{ii} = \frac{i}{6}$ and $p_{i,i+1} = 1 - \frac{i}{6}$ for $i = 0,1,2,\dots,6$

All other $p_{ij} = 0$.

$P(\text{more than } m \text{ throws needed}) = 1 - p_{06}^{(m)}$.

3.8 Let $X_n =$ Joe's money after n runs. State space $I = \{10i\colon i = 0,1,\dots,5\} \cup \{5i\colon i = 11,12,\dots,40\}$.

Note that state 200 means "200 or more". Let 0 and 200 be absorbing states. The transition probabilities become

$p_{10i,10(i-1)} = 0.60$ and $p_{10i,10(i+1)} = 0.25$, $i = 1,\dots,5$
$p_{5i,5(i-1)} = 0.60$ and $p_{5i,5(i+1)} = 0.25$ and $p_{5_i,5(i+2)} = 0.15$, $i = 11,12,\dots,40$
$p_{00} = p_{200,200} = 1$

The other $p_{ij} = 0$.

Let $\mu_s =$ expected number of bets to state 0 or 200 from state s.

$\mu_0 = \mu_{200} = 0$

$\mu_5 = 1 + 0.25\mu_{10} + 0.15\mu_{15}$

$\mu_{10} = 1 + 0.60\mu_5 + 0.25\mu_{15} + 0.15\mu_{20}$

...

Expected number of bets when starting in 100 is $\mu_{100} = 212.29$. Similarly, using a system of linear equations gives

$$P(\text{reaching } 200 | X_0 = 100) = 0.2132$$

3.10 Uniform means that for N states $\pi_i = 1/N$. If π_i is uniform we get

$$\frac{1}{N} = \sum_{i \in I} p_{ij} \frac{1}{N} \Leftrightarrow \sum_{i \in I} p_{ij} = 1$$

i.e. if the columns sum to 1, which they do.

3.11 Let $I = \{\text{win,loose}\} = \{1,2\}$. Transition matrix

$$P = \begin{pmatrix} 0.25 & 0.75 \\ 0.50 & 0.50 \end{pmatrix}$$

The equilibrium distribution is $\{\pi_1, \pi_2\} = \{0.4, 0.6\}$. The long-run net amount won is

$$0.4 * 2.50 - 1 = 0$$

The game is fair!

3.12 The long-run average cost equals 17.77.

3.13 (a) The long-run fraction of games won is 0.4957.

(c) The long-run fraction of games won is 0.5073.

3.15 (a) Markov chain $\{X_n\} = \left\{X_n^{(1)}, X_n^{(2)}\right\}$ where

$X_n^{(i)}$ = the age in days at the beginning of day n of component $i = 1,2$.

Let M denote the failure state. The state space is

$$I = \{(i_1, i_2): 1 \le i_1 \le R \text{ or } i_k = M \text{ for } k = 1,2\}$$

Transition probabilities for $1 \le i_1, i_2 < R$

$$p_{(i_1,i_2),(j_1,j_2)} = \begin{cases} (1 - r_{i_1})(1 - r_{i_2}) \; j_1 = i_1 + 1, \; j_2 = i_2 + 1 \\ (1 - r_{i_1}) r_{i_2} \; j_1 = i_1 + 1, j_2 = M \\ r_{i_1}(1 - r_{i_2}) \; j_1 = M, j_2 = i_2 + 1 \\ r_{i_1} r_{i_2} \; j_1 = M, j_2 = M \end{cases}$$

where r_j = probability that component of age j fails the next day. Similar for $i_1 = R$ and $1 \le i_2 < r$ with $j_1 = 1$ above. For $i_1 = R$ and $i_2 > r$, take $j_1 = j_2 = 1$. Vice versa for $1 \le i_1 \le r$ and $i_2 = R$.

(b) Let K_1 and K_2 denote the costs of replacing 1 and 2 components respectively. The long run average cost becomes

$$K(r,R) = K_1\left(\sum_{i_1=1}^{r-1}\big(\pi(i_1,R)+\pi(i_1,M)\big)+\sum_{i_2=1}^{r-1}\big(\pi(M,i_2)+\pi(R,i_2)\big)\right)$$
$$+K_2\left(\sum_{i_1=r}^{M}\big(\pi(i_1,R)+\pi(i_1,M)\big)+\sum_{i_2=r}^{M}\big(\pi(M,i_2)+\pi(R,i_2)\big)-\pi(M,R)\right.$$
$$\left.-\pi(R,M)-\pi(R,R)-\pi(M,M)\right)$$

3.17 Let X_n = number of messages present at time n. Let

$$\delta_n = \begin{cases} 0 & \text{if gate is closed at } n \\ 1 & \text{otherwise} \end{cases}$$

Then $\{(X_n, \delta_n)\}$ is a Markov chain with state space

$$I = \{(i,1): i = 0,1,\dots,R-1\} \cup \{(i,0): i = r+1,\dots,R\}$$

Transition probabilities

$p_{(i,1),(j,1)} = e^{-\lambda}\frac{\lambda^{j-i+1}}{(j-i+1)!}\ \ i,j = 0,1,\dots,R-1$

$p_{(i,1),(R,0)} = \sum_{k=R-i+1}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}\ \ i = 1,\dots,R-1$

$p_{(r+1,0),(j,1)} = e^{-\lambda}\frac{\lambda^{j-r}}{(j-r)!}$

$p_{(0,1),(R,0)} = \sum_{k=R}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}$

$p_{(r+1,0),(R,0)} = \sum_{k=R-r}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}$

$p_{(i,0)(i-1,0)} = 1,\ \ i = r+2,\dots,R$

(b) Long-run fraction of lost messages

$$L(r,R) = \sum_{i=0}^{R-1}\pi_{(i,1)}\sum_{k=R-i+1}^{\infty}(k-R+i-1)e^{-\lambda}\frac{\lambda^k}{k!}+\pi_{(r+1,0)}\sum_{k=R-r}^{\infty}(k-R+r)\,e^{-\lambda}\frac{\lambda^k}{k!}$$
$$+\sum_{i=r+2}^{R}\pi_{(i,0)}\cdot 1\cdot\lambda+\pi_{(0,1)}\sum_{k=R}^{\infty}(k-R)e^{-\lambda}\frac{\lambda^k}{k!}$$

3.19 Recall that Erlang(r,μ) can be seen as the sum of r independent $\mathrm{Exp}(\mu)$ subtasks.

Let X_n = number of subtasks present just before n. State space $I = \{0,1,2,\dots\}$.

Transition probabilities for $i = 0,1,\dots$ and $j = 1,2,\dots,i+r$

$$p_{ij} = e^{-\mu}\frac{\mu^{i+r-j}}{(i+r-j)!}$$

4.1 $X_1(t)$ = number of waiting passengers at t

$$X_2(t) = \begin{cases} 1 & \text{sheroot is present at time } t \\ 0 & \text{otherwise} \end{cases}$$

State space $I = \{(i,0): i = 0,1,\dots,7\} \cup \{(i,1): i = 0,1,\dots,6\}$.

4.2 Let

$$X_i = \begin{cases} 0 & \text{if unit } i \text{ is free at time } t \\ 1 & \text{if unit } i \text{ services district } 1 \\ 2 & \text{if unit } i \text{ services district } 2 \end{cases}$$

$\{(X_1(t), X_2(t))\}$ is a Markov chain with state space $I = \{(i,j): i,j = 0,1,2\}$

4.3 Let $\{X(t)\}$ be a continuous-time Markov chain with state space $I = \{(0,0),(0,1),(1,1),(b,1)\}$ where

$(0,0)$ = both stations free
$(0,1)$ = station 1 is free, station 2 busy
$(1,1)$ = both stations busy
$(b,1)$ = station 1 is blocked, station 2 is busy

4.4 (a) Let $X(t)$ = number of cars present at t. State space $I = \{0,1,2,3,4\}$.

(b) The equilibrium probabilities are given by

$$\begin{aligned} p_0 &= 0.3839 \\ p_1 &= 0.2559 \\ p_2 &= 0.1706 \\ p_3 &= 0.1137 \\ p_4 &= 0.0758 \end{aligned}$$

4.5 Let $X(t)$ = state of production hall at time t, with state space

$$I = \{(0,0),(1,0),(0,1),(1,1)\}$$

where

$(0,0)$ = both machines are idle
$(1,0)$ = the fast machine is busy, the slow machine is idle
$(0,1)$ = the fast machine is idle, the slow machine is busy
$(1,1)$ = both machines are busy

(b) The equilibrium equations are given by

$$\begin{aligned}
&\lambda p(0,0) = \mu_1 p(1,0) + \mu_2 p(0,1)\\
&(\lambda + \mu_1) p(1,0) = \lambda p(0,0) + \mu_2 p(1,1)\\
&(\lambda + \mu_2) p(0,1) = \mu_1 p(1,1)\\
&(\mu_1 + \mu_2) p(1,1) = \lambda p(1,0) + \lambda p(0,1)\\
&p(0,0) + p(0,1) + p(1,0) + p(1,1) = 1
\end{aligned}$$

The long-run fraction of time that the fast machine is used is given by $p(1,0) + p(1,1)$ and the slow machine by $p(0,1) + p(1,1)$. The long-run fraction of incoming orders that are lost equals $p(1,1)$.

4.6 Let $X(t)$ denote the state of the system at time t. There are 13 states.

$(0,0) =$ a taxi is waiting but no customers are present

$(i,k) = i$ customers are waiting at the station, no taxi is there and the taxi took k customers last time, $i = 0,1,2,3,\ k = 1,2,3$.

The equilibrium equations will be on the form

$$\lambda p(0,1) = \mu_1 p(0,1) + \mu_2 p(0,2) + \mu_3 p(0,3)$$

The long-run fraction of taxis waiting is $\lambda p(0,0)$. The long-run fraction of customers that potentially goes elsewhere is $p(3,1) + p(3,2) + p(3,3)$.

4.7 Let $X_1(t) =$ number of trailers present at t

$$X_2(t) = \begin{cases} 1 & \text{if unloader in finishing process} \\ 0 & \text{otherwise} \end{cases}$$

The process has state space

$$I = \{(i,j): i = 0,1,\dots,N, j = 1,2\}$$

Equilibrium equations

$$\begin{aligned}
&N\lambda p(0,0) = \mu_2 p(0,1)\\
&(\mu_1 + (N-i)\lambda) p(i,0) = (N-i+1)\lambda p(i-1,0) + \mu_2 p(i,1), \quad 1 \le i \le N-1\\
&(\mu_2 + (N-i)\lambda) p(i,1) = (N-i+1)\lambda p(i-1,1) + \mu_1 p(i+1,0), \qquad 0 \le i \le N-1\\
&\mu_1 p(N,0) = \lambda p(N-1,0) + \mu_2 p(N,1)
\end{aligned}$$

4.8 Let $X_1(t) =$ number of messages in the system at t

and $X_2(t) = \begin{cases} 1 & \text{if gate is open at } t \\ 0 & \text{otherwise} \end{cases}$

State space

$$I = \{(i,1): i = 0,1,\dots,R\} \cup \{(i,0): i = r+1,\dots,R+1\}$$

The long-run fraction of time the channel is idle is $p(0,1)$.

The long-run fraction of messages blocked is $\sum_{i=r+1}^{R+1} p(i,0)$.

The long-run average number of messages waiting to be transmitted is

$$\sum_{i=1}^{R}(i-1)p(i,1) + \sum_{i=r+1}^{R+1}(i-1)p(i,0)$$

4.9 $X(t) =$ number of customers at t. State space $I = \{0,1,\dots\}$.

Equate the rate out of set $\{i, i+1, \dots\}$ to the rate into the set and get recurrence relation

$$\mu p_i = \frac{\lambda}{i} p_{i-1}$$

which gives

$$p_i = \frac{(\lambda/\mu)^i}{i!} p_0$$

and thus

$$p_0 = e^{-\lambda/\mu}$$

$$p_i = e^{-\lambda/\mu}\frac{(\lambda/\mu)^i}{i!}, \quad i = 1,2,\dots$$

The long-run fraction of persons that join the queue is

$$\sum_{i=0}^{\infty} p_i \frac{1}{i+1} = \frac{\mu}{\lambda} - \frac{\mu}{\lambda} e^{-\lambda/\mu}$$

The long-run average number of persons served per time unit is

$$\lambda\left(\frac{\mu}{\lambda} - \frac{\mu}{\lambda} e^{-\lambda/\mu}\right) = \mu(1 - p_0)$$

4.10 (a) Equilibrium equations

$$\begin{aligned}
&(\lambda+\mu)p(0,0) = \mu p(7,0) + \lambda p(6,1)\\
&(\lambda+\mu)p(i,0) = \lambda p(i-1,0) \quad i = 1,\dots,6\\
&\mu p(7,0) = \lambda p(6,0)\\
&\lambda p(i,1) = \lambda p(i-1,1) \quad i = 1,\dots,6
\end{aligned}$$

The long-run fraction of potential customers lost is $p(7,0)$.

(b) Now the states are

$(0,0) =$ no sheroot is in and none is waiting
$(i,1) = i$ passengers are waiting and the sheroot is in, $i = 0,\dots,6$

Equilibrium equations

$$\mu p(0,0) = \lambda p(6,1)$$
$$\lambda p(0,1) = \mu p(0,0)$$
$$\lambda p(i,1) = \lambda p(i-1,1) \quad i = 1, \dots, 6$$

The long-run fraction of potential customers lost is now $p(0,0)$.

4.12 $X(t) = \text{stock at } t$. State space $I = \{1, \dots, Q\}$.

Equilibrium equations

$$(\lambda + \mu)p_1 = (\lambda + Q\mu)p_Q$$
$$(\lambda + i\mu)p_i = (\lambda + (i+1)\mu)p_{i+1} \quad i = 1, \dots, Q-1$$

The long-run average stock is $\sum_{i=1}^{Q} ip_i$.

The long-run average number of orders per time-unit = long-run average number of transitions from 1 to Q = $(\lambda + \mu)p_1$.

4.16 Let $X(t) = \text{number of units broken at } t$. State space $I = \{0,1, \dots, s+1\}$.

Construct a modified Markov chain with absorbing state $a = s + 1$. The state space is the same but the leaving rates become

$$\nu_i^* = \begin{cases} \nu_i & \text{for } i = 0,1, \dots, s \\ 0 & \text{for } i = s+1 \end{cases}$$

and

$$q_{ij}^* = \begin{cases} q_{ij} & \text{if } i = 0,1, \dots, s, \ \ j = 0,1, \dots, s+1, i \neq j \\ 0 & \text{if } i = s+1 \end{cases}$$

Find $p_{ij}^*(t)$ using the uniformization method where $p_{s+1,s+1}^* = 1$

5.1 (a) $X(t) = \text{number of customers present at } t$, state space $I = \{0,1, \dots, c+N\}$

For $\{i, i+1, \dots, c+N\}$ the recursive relation

$$\min(i,c)\mu p_i = \lambda p_{i-1}$$

which gives

$$p_i = \begin{cases} \dfrac{(\lambda/\mu)^\wedge i}{i!} p_0 & \text{if } i = 1, \dots, c-1 \\ \left(\dfrac{\lambda}{c\mu}\right)^{i-c+1} p_{c-1} & \text{if } i = c, \dots, c+N \end{cases}$$

(b) Error in book! Should be

$$W_q(x) = 1 - \frac{1}{1 - p_{c+N}} \sum_{j=c}^{c+N} p_j \sum_{k=0}^{j-c} e^{-c\mu x} \frac{(c\mu x)^k}{k!}$$

5.2 (a) $X(t) = \text{service requests at } t$.

For $\{i, i+1, \ldots, N\}$

$$\min(i, c)\mu p_i = (N - i + 1)\nu p_{i-1}$$

which gives

$$p_i = \begin{cases} \binom{N}{i}\left(\frac{\nu}{\mu}\right)^i p_0 & \text{if } i = 1, \ldots, c-1 \\ \frac{(N-c+1)!}{(N-i-1)!}\rho^{i-c+1} p_{c-1} & \text{if } i = c, \ldots, N \end{cases}$$

5.8 $\{p_j, j \in I\}$ satisfy the equilibrium equations. Substitute the p_i^A in these equations using the given expression and check if it holds.

5.9 $X(t) = \text{number of containers present at } t, I = \{0,1, \ldots, L\}.$

For $\{i, i+1, \ldots, L\}$

$$i\mu p_i = \lambda p_{i-1}$$

which gives

$$p_i = \frac{e^{-\lambda/\mu}\left(\frac{\lambda}{\mu}\right)^i / i!}{\sum_{k=0}^{c} e^{-\lambda/\mu}\left(\frac{\lambda}{\mu}\right)^k / k!}$$

With $\lambda = 1$ and $1/\mu = 10$ we get that $L \geq 18$ since

$$p_L = \begin{cases} 0.0129 & \text{for } L = 17 \\ 0.00714 & \text{for } L = 18 \end{cases}$$

5.10 $X_i(t) = \text{number of cars at } t \text{ of type } i = 1,2$

$\{X(t)\} = \{(X_1(t), X_2(t))\}$ has state space $I = \{(i_1, i_2): i_1, i_2 = 0,1, \ldots, c, i_1 + i_2 \leq c\}$

$$p(i_1, i_2) = \frac{\frac{\left(\frac{\lambda_1}{\mu_1}\right)^{i_1}\left(\frac{\lambda_2}{\mu_2}\right)^{i_2}}{i_1!\, i_2!}}{\sum_{k_1+k_2 \leq c} \frac{\left(\frac{\lambda_1}{\mu_1}\right)^{k_1}\left(\frac{\lambda_2}{\mu_2}\right)^{k_2}}{k_1!\, k_2!}}$$

$$P_{\text{loss}} = \sum_{k_1+k_2 \leq c} p(k_1, k_2)$$

5.14 $X(t) = \text{number of working components at } t, I = \{0,1, \ldots, c\}$

For $\{i, i+1, \ldots, c\}$

$$i\alpha p_i = \beta p_{i-1}$$

giving the truncated Poisson as usual.

The long-run fraction of time the system is down is then $= p_0$.

(b) No, due to the insensitivity property.