Chapter 1

Robustness and Distribution Assumptions

1.1 Distribution Assumptions in Statistics

1.1.1 Introduction

In statistics, one often works with model assumptions, i.e., one assumes that data follow a certain model. Then one makes use of methodology that is based on the model assumptions.

With the above setup, choosing the methodology can be a quite delicate issue, since the performance of many methods may be very sensitive to whether the model assumption hold or not. For some methods, even very small deviations from the model may result in poor performance.

Methods that perform well, even when there are some (more or less) minor deviation from the model assumption, are called *robust*.

1.1.2 Distribution Assumptions in Statistics

Let X be a real-valued random variable (r.v.), which is assumed to have a certain specific distribution function $F : \mathbb{R} \to [0, 1]$. Here F is allowed to depend on a *parameter* $\theta \in \mathbb{R}^m$, so that the distribution can be written as

$$\mathbf{P}\{X \leq x\} = F(x; \theta) \qquad \text{for} \quad x \in \mathbb{R}.$$

The parameter θ is assumed to have a certain specific value, which is normally not known.

Example 1.1. *The random variable X is assumed to have normal $N(\mu, \sigma^2)$-distribution for some (unknown) selection of the parameter $\theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}^+$.*

The above-mentioned type of scenario, or variants there of, are the frameworks for *parametric statistic methods*. One example, where the method uses a distributional assumption in a crucial manner, is analysis of variance, which assumes normal distribution, and it is not applicable when that assumption is violated.

Observe that, in practice, one can usually not uncritically accept assumptions on the distribution as valid. Hence it is important to be able to determine if the data really comes from an assumed distribution $F(\cdot; \theta)$, for some value of the parameter θ.

Let $X_1, \ldots, X_n$ be a *random sample* of X, i.e., independent random variables with the same distribution as X (which is $F(\cdot;\theta)$ if the assumption on the distribution holds). For the above mentioned reasons, it is often of importance to determine whether the distribution of X really is $F(\cdot;\theta)$. This cannot be done in a completely precise manner, as we have randomness.

In fact, to test the distribution assumption $F(\cdot;\theta)$, one has to use some statistical test, which hopefully, with a large probability of being correct, can tell whether the data obeys the assumption.

1.2 Parameter and Density Estimation

1.2.1 Maximum Likelihood Estimation

Let $x_1, \ldots, x_n$ be a random sample from a r.v. X (assumed) having density $f_X(x;\theta)$. The *likelihood function* is defined as

$$L(\theta; x_1, \ldots, x_n) = f(x_1, \ldots, x_n, \theta) = \prod_{i=1}^{n} f_X(x_i;\theta).$$

Note that it is a function of the parameter θ, the values $x_1, \ldots, x_n$ come from our observations! The maximum likelihood (ML) estimator of θ is

$$\hat{\theta} = \arg\max_{\theta \in \mathbb{R}^m} L(\theta; x_1, \ldots, x_n)$$

It is often easier to regard the logarithm of the likelihood function, i.e. $l(\theta, x_1, \ldots, x_n) = \log L(\theta, x_1, \ldots, x_n)$ and maximize this instead.

Example 1.2. *Let $x_1, \ldots, x_n$ be a random sample from a r.v. which is Exp(λ)-distributed, i.e. $f_X(x;\lambda) = \lambda e^{-\lambda x}$. This means that the likelihood function becomes*

$$L(\lambda; x_1, \ldots, x_n) = \lambda^n \prod_{i=1}^{n} e^{-\lambda x_i}.$$

In this case it is analytically more tractable to regard the log-likelihood function, i.e.

$$l(\lambda; x_1, \ldots, x_n) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i.$$

The ML-estimator is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}.$$

1.2.2 Kernel Density Estimation

Let $Y_1, \ldots, Y_n$ be a random sample from a r.v. having unknown density $f_Y(y;\theta)$. A kernel estimate of $f_Y(y;\theta)$ is

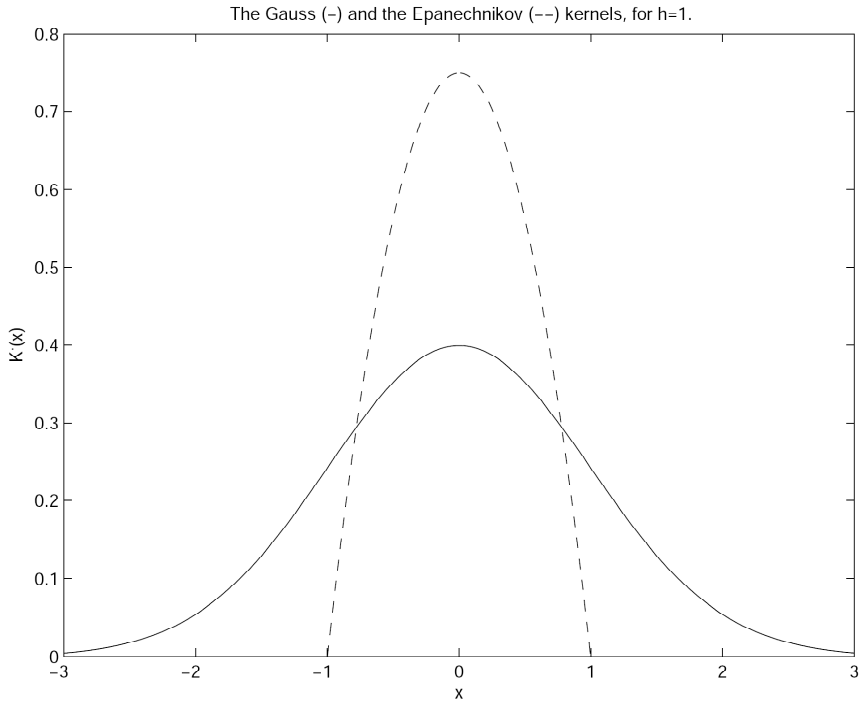
$$\hat{f}(y;h) = \frac{1}{n} \sum_{i=1}^{n} K_h(y - Y_i).$$

Here $K_h(t) = \frac{1}{h}K(\frac{t}{h})$, where $K(s)$ is a function satisfying $\int K(s)\,ds = 1$, which we call a *kernel* and h is called the *bandwidth*. Two widely used kernels are the *Gauss* and the *Epanechinov* kernel. They are defined respectively as

$$K^G(s) = \frac{1}{\sqrt{2\pi}}e^{-s^2/2}, \tag{1.1}$$

$$K^E(s) = \frac{3}{4}(1-s^2)\mathbf{1}_{\{|s|<1\}}. \tag{1.2}$$

Figure 1.1: The Gauss (-) and the Epanechnikov(- -) kernels (h=1).



Example 1.3. *Consider a r.v. Y which is a mixture of normal distributions i.e.*

$$Y = UX + (1-X)V,$$

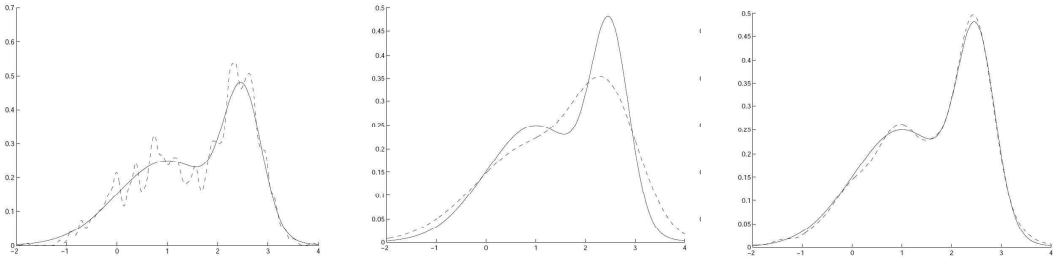
where $U\sim N(1,1)$, $V\sim N(\frac{5}{2},(\frac{3}{8})^2)$ and $X\sim Ber(\frac{5}{8})$ and independent. The density of Y is

$$f_Y(y) = \frac{5}{8}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-1)^2}{2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\frac{5}{2})^2}{2(\frac{3}{8})^2}}$$

In Figure 1.2 we see kernel density estimates, using three different bandwidths of $f_Y(y)$ based on a simulated random sample of size n=1000.

As we see in example 1.3, choosing the bandwidth is an important issue. One way of doing this is to do a *least squares cross validation*, which means to choose the h that

Figure 1.2: $f_Y(y)$ (-) and kernel density estimates of $f_Y(y)$ (- -). (a): h=0.05, (b): h=0.5, (c): h=0.2.



(a) (b) (c)

minimizes

$$LSCV(h) = \int \hat{f}(x;h)^2\,dx - \frac{2}{n}\sum_{i=1}^{n} \hat{f}_{-i}(X_i;h),$$

where

$$\hat{f}_{-i}(x;h) = \frac{1}{n-1}\sum_{j\neq i}^{n} K_h(x - X_j),$$

i.e. doing the estimation when observation i is removed.

1.3 Test of Distribution Assumptions

1.3.1 Graphical Test of the Distribution Assumptions

We start with stating some facts that will be of importance to us:

Let X be a random variable that have a continuous distribution function F. Then the random variable $F(X)$ has a uniform distribution over $[0,1]$. To see this, just notice that

$$\mathbf{P}\{F(X) < x\} = \mathbf{P}\{X < F^{-1}(x)\} = F(F^{-1}(x)) = x \qquad \text{for} \quad x \in [0,1]^1.$$

The above fact is very useful, because it says that if we have a sample of random variables, and we perform a transformation so that they become uniformly distributed over $[0,1]$, then the transformation should (more or less) be the distribution function!

Now, as a direct consequence of the *Glivenko-Cantelli theorem*[2](see Chapter 3), we have the following theorem:

[1]Here F^{-1} is a generalized inverse, if F is non-invertible.

[2]Glivenko-Cantelli says that the empirical distribution of a sample of a random variable converges uniformly to the distribution of the random variable, as sample size tends to infinity.

Theorem 1.1. *If the sample $X_1, \ldots, X_n$ has distribution function $F(\cdot;\theta)$, then for the ordered sample $X_{(1)} \leq \ldots \leq X_{(n)}$, we have*

$$\lim_{n\to\infty} \max_{1\leq i\leq n} \left|(i-0.5)/n - F(X_{(i)};\theta)\right| = 0.$$

Now, if the assumption that the sample $X_1, \ldots, X_n$ has the distribution function $F(\cdot;\theta)$ is correct, then, according to Theorem 1.1,

$$\max_{1\leq i\leq n} \left|(i-0.5)/n - F(X_{(i)};\theta)\right| \approx 0 \qquad \text{for large} \quad n.$$

Consequently, a plot of the sequence of pairs

$$\left\{\left((i-0.5)/n, F(X_{(i)};\theta)\right)\right\}_{i=1}^{n},$$

a so-called *pp-plot*, is approximately a 45°-line. The same is then true for a so-called *qq-plot* of the sequence

$$\left\{\left(X_{(i)}, F^{-1}((i-0.5)/n;\theta)\right)\right\}_{i=1}^{n}.$$

A systematic discrepancy of a *pp*-plot or *qq*-plot from a 45°-line indicates that the $F(\cdot;\theta)$-assumption is not true. Notice that, because of randomness, these plots never become completely straight-lined, for a finite sample size n, even when the $F(\cdot;\theta)$-assumption holds, but always display a certain random variation around the 45°-line. The larger n, the smaller that random variation becomes.

When the $F(\cdot;\theta)$-assumption is false, an additional systematic discrepancy from the 45°-line occurs, resulting in an (in some sense) curved plot.

Normally, the value of the parameter θ is not known, and hence must be estimated by an estimator $\hat{\theta}$. Supposing that $F(\cdot;\theta)$ is a continuous function of θ, and that the estimator $\hat{\theta}$ is *consistent*, i.e., that it converges to θ when $n \to \infty$, the following *pp*- and *qq*-plots would be approximate 45°-lines

$$\left\{\left((i-0.5)/n, F(X_{(i)};\hat{\theta})\right)\right\}_{i=1}^{n} \qquad \text{and} \qquad \left\{\left(X_{(i)}, F^{-1}((i-0.5)/n;\hat{\theta})\right)\right\}_{i=1}^{n},$$

when the $F(\cdot;\theta)$-assumption holds.

The decision whether a *pp*- or *qq*-plot displays systematic discrepancy, or only random variation discrepancy, from a 45°-line, is conveniently done by means of a comparison with a *reference plot*, without systematic discrepancy. This in turn, can be done by generating a sample $Y_1, \ldots, Y_n$ from a random variable Y that really has the distribution function $F(\cdot;\theta)$, or $F(\cdot;\hat{\theta})$ if θ is unknown and estimated, so that the *pp*-plot

$$\left\{\left((i-0.5)/n, F(Y_{(i)};\theta)\right)\right\}_{i=1}^{n} \qquad \text{or} \qquad \left\{\left((i-0.5)/n, F(Y_{(i)};\hat{\theta})\right)\right\}_{i=1}^{n},$$

and the *qq*-plot

$$\left\{\left(Y_{(i)}, F^{-1}((i-0.5)/n;\theta)\right)\right\}_{i=1}^{n} \qquad \text{or} \qquad \left\{\left(Y_{(i)}, F^{-1}((i-0.5)/n;\hat{\theta})\right)\right\}_{i=1}^{n}$$

display only random variation discrepancy from a 45°-line.

Of course, systematic variations from a 45°-line can be hidden by large random variations, when the sample size n is small in relation to the that the systematic variation. A non-significant *pp*- or *qq*-plot, without clear systematic variations from a 45°-line, does not necessarily imply that the $F(\theta;\cdot)$-assumption is true[3]. However, one can conclude that the random variation of the data material is similar to the variations of true $F(\theta;\cdot)$-data. This in turn, hopefully, should be enough for making practical use of the $F(\theta;\cdot)$ assumption, at least with robust methodology.

1.3.2 Statistical Test of Distribution Assumptions

In the previous section we described how to test a distribution assumption qualitatively, by a graphical procedure. However, to do it quantitatively, we have to employ a test, which produces a statistic, and hence gives us a p-value, that may be significant or non-significant, in turn.

Chi-Square Goodness-of-Fit Test

With the *chi-square test*, given an assumed continuous or discrete distribution $F(\cdot;\theta)$ for a sample, one can assign probabilities that a random variable has a value within an interval, or a so called *bin*. Quite obviously, the actual value of the chi-square test statistic will dependent on how the data is binned.

One disadvantage with the chi-square test, is that it is an *asymptotic test* (rather than exact one), i.e., it requires a large enough sample size in order for the chi-square approximation to be valid. As a rule of thumb, each bin should contain at least 5 observations from the sample.

The *chi-square test statistic* of k bins is given by

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i},$$

where O_i is the observed frequency for bin i, i.e., the number of observations that lies in the bin $(l_i, u_i]$, and

$$E_i = n[F(u_i;\hat{\theta}) - F(l_i;\hat{\theta})]$$

is the expected frequency of bin i, with F denoting the assumed distribution function. Here n is the sample size, as before, and $l_1 < u_1 < l_2 < u_2 \leq \ldots \leq l_k < u_k$, with k being the number of bins.

Under the null hypothesis, that the F-assumption is true, the test statistic χ^2 follows, approximately, a chi-square distribution, with $k - c$ degrees of freedom, where c is the number of estimated parameters.

Kolmogorov-Smirnov Goodness-of-Fit Test

The *Kolmogorov-Smirnov test* is applicable when assuming a continuous distribution F for the sample $X_1, \ldots, X_n$. The test statistic is given by[4]

$$D = \max_{1\leq i\leq n} \left| F(X_{(i)};\hat{\theta}) - \frac{i}{n} \right|,$$

[3] In the same fashion, a non-significant outcome of a statistical hypothesis test does not necessarily imply that the null hypothesis is true.

[4] We have previously used $(i - 0.5)/n$ instead of i/n: This choice is really a matter of taste, and of no practical importance.

were $X_{(1)} \leq \ldots \leq X_{(n)}$ is the ordered sample. As before, all unknown parameters for F have to be estimated.

Observe that the Kolmogorov-Smirnov statistic D is a measure of how much a *pp*-plot deviates from a 45°-line.

To calculate the p-value for D, one makes use of the fact that $\sqrt{n}D$ is asymptotically *Kolmogorov distributed*, under the null hypothesis. The distribution function of the Kolmogorov distribution is given by

$$Q(x) = 1 - 2\sum_{i=1}^{\infty}(-1)^{i-1}\mathrm{e}^{-2i^2x^2}.$$

In practice, there is seldom any need to do manual computations with the Kolmogorov distribution, as the computations are handled by statistical programs.

Some extension of the Kolmogorov-Smirnov test has been made, to emphasize certain regions of values. One important example, is the *Kuiper test*, with test statistic

$$K = \max_{1\leq i\leq n}\left(F(X_{(i)};\hat{\theta}) - \frac{i}{n}\right) + \max_{1\leq i\leq n}\left(\frac{i}{n} - F(X_{(i)};\hat{\theta})\right).$$

A Kuiper test emphasizes the importance of the *tails*, i.e., the smallest and largest observations. This is of importance in applications to assessment of risk.

1.4 Robust Estimation

One of the most natural illustrations of *robust estimation* techniques, is the estimation of a location parameter, of a continuous symmetric distribution: Assume that we have a sample $X_1, \ldots, X_n$ from a distribution function of the form $F(x, \theta) = F(x-\theta)$, where $\theta \in \mathbb{R}$. Here θ is called a *location parameter*.

Example 1.4. *If F is a normal distribution, then θ coincides with the expected value and the median. Further, the* sample mean

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is a good estimator of θ.

Example 1.5. *If F is Cauchy distributed, with probability density function*

$$f_X(x;\theta) = \frac{1}{\pi(1+(x-\theta)^2)},$$

then the sample mean is not a good estimator of the location parameter θ. The reason for this is that the Cauchy distribution allows, with a large probability very large values, and does in fact not even have a well-defined (finite) expected value. This means that the sample may display some extremely small or large "non-typical" values, so called outliers, *and a few such may heavily influence the sample mean, so that it deviates significantly from θ.*

To avoid the problem indicated in Example 1.5, one may, for example, replace the sample mean with the *sample median*, as estimator of the location parameter: The

latter does not display the sensitivity to outliers, as does the former. In other words, the sample median is a *robust estimator*.

Robustness can, of course, be defined in mathematical terms. That description usually is based on the *influence function*, which measures the sensitivity to outliers. However, this subject matter goes beyond the scope of this course.

An intuitive way, to view the issue of the robustness of an estimator, is to look at the *breakdown point*. This is the largest percentage of data points that can be changed arbitrarily, without causing undue influence on the estimator.

Example 1.6. *The sample median has* 50% *breakdown. This is illustrated by the fact that, for a sample of* 100 *ordered data, the first* 49 *can be changed arbitrarily, as long as their values stay smaller than the* 50*:th smallest observation, without affecting the value of the sample median at all.*

The sample mean is not a robust estimator, because changing the value of a single observation may heavily influence the value of the sample mean. This means that the sample mean has breakdown point 0%*.*

In practice, the choice of estimator is a trade-off between robustness and *efficiency*, as very robust estimators tends to be inefficient, i.e., they do not make full use of the data material.

It should be noted that robustness is related to the concept of *non-parametric statistics*, i.e., statistical methodology that do not rely on distribution assumptions.

Naturally, there exist robust estimators for other things than location parameters. Many robust estimators can be grouped into one of the following three different classes:

M-estimators, which are based on maximum-likelihood arguments: These estimators are commonly used in the fitting of models and parameter estimations.

L-estimators, which are linear combination of order statistics. The sample median and the *trimmed mean* (that will be presented in the laboration below) are examples on this.

R-estimators, which are based on rank tests: The Kolmogorov-Smirnov statistic is an example on this.

1.5 Some tips on Matlab

Matlab a very common mathematical programme package, that is quite easy to use. The program has many built in functions, and as it is widely spread, one can search the internet for additional free software libraries.

When using Matlab, one should get used to employ its built in help function. That help function can be reach, simply by writing `help subject`. For example, `help stats` gives a list of all function in the statistical toolbox, while `help hist` gives help on the function `hist`. In addition, one can make use of the commands `helpdesk` and `helpwin`, for easy access to help.

When using Matlab, one should make use of an editor, rather than writing commands directly in the Matlab window. The reason for this is that, when writing long programs, it is convenient to have everything in "one place", easily accessible and editable.

One may use the built in editor of Matlab. One can also start Matlab with `matlab -nojvm`, and make use of the *emacs* editor: To run a Matlab program, written with

emacs, save the code to a m-file, called `foo.m`, say.[5] Notice that one have to save the file to the directory that Matlab is run from, in order to avoid specifications of paths.

When plotting, it is nice to have colourfull graphs. However, when you print those graphs, they will usually be black and white. Hence it is a good practice to design the graphs, so that they work well also in black and white. One example of this, is to use solid, dotted and dashed lines, etc. to distinguish different graphs, rather than different colours only. It can be quite useful, and is simple, to add a Matlab *legend* to a plot.

Sometimes, Matlab can be really slow especially when one uses loops. Always try to avoid loops, when there is an alternative! For example, sometimes a (slow) loop can be replaced with a (quicker) vector multiplication.

1.6 Laboration

1.6.1 Test of Distribution Assumptions

The file *ibm.txt* contains the stock prizes S_t of the IBM stock for every trading day t between 1964-2004. The corresponding *logreturns* X_t are defined as the logincremets

$$X_t = \log(S_t/S_{t-1}) = \log(S_t) - \log(S_{t-1}).$$

It is a common assumption, in the world of mathematical finance, that the logreturns are normal distributed.

1. *Load ibm.txt into Matlab with the command* `load ibm.txt`.

   Before doing this, do not forget to put a % on the beginning of first line of data file ibm.txt, with the emacs editor. This have to be done, because the first line of the data file are character headers, with variable names. And Matlab cannot read that text, so that Matlab has to be instructed to neglect it.

   Now, in the variable `ibm` you have the data.

2. *Write a program to compute the* 1000 *first logreturns from column number* 7*, which contains the stock prizes* S_t*.*

   Observe that one cannot divide with 0. This means that one has to neglect such data, if they exist. Also, modify the indexing of the vector, so that the first observation has index 1, the second 2, and so on, rather than their actual dates.

   Now, putting the calculated logreturns in the variable `logret`, one can plot the stock prizes with the command `plot(exp(cumsum(logret)))`.

3. *Derive the ML-estimates of* μ *and* σ^2*, if we assume the logreturns to be independent and* $N(\mu, \sigma^2)$*-distributed. Do a kernel density estimate and plot it together with the estimated normal density.*

4. *Produce a qq- or pp-plot, using the normal distribution assumption. Test the normal distribution assumption graphically, by means of compare it with a reference plot, based on a sample with normal distributed random variables.*

5. *Perform a quantitative test to investigate if the logreturns are normal distributed.*

[5] If a Matlab m-file is a function, then one will not have access to objects created within that function, as they are well-defined "locally" only, inside that function.

1.6.2 Effect of Distribution Assumption

Assume that $X_1, \ldots, X_n$ is a random sample form a normal distribution with parameters μ and σ. Now, an estimator for the variance σ^2, when μ is not known, is the sample variance

$$s_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2,$$

where $\bar{X}$ is the sample mean.

Now, if the normal distribution assumption holds, then s_X^2 is a sum of n squared normal distributed random variables, so that

$$\frac{(n-1)s_X^2}{\sigma^2}$$

is chi-square distributed, with $n-1$ degrees of freedom. Based on this information, we can calculate a confidence interval for σ^2:

1. *Generate* 100 *normal distributed random variables with parameters* $\mu = 0$ *and* $\sigma = 1$. *Calculate a confidence interval for* σ^2, *with confidence level* 0.95.

   Repeat the above procedure 1000 *times, and count the number of intervals which contains* σ^2. *Also, calculate the average width of the intervals.*

   How does this compare with what you expect from the confidence level being 0.95*?*

2. *Now do the same thing all over again, but this time for a Student-t distributed random sample* $X_1, \ldots, X_n$, *with a parameter* $\nu > 2$.*(That is calculate the intervals under the assumption of normal distribution, but using the Student-t random sample)*

   Remember that the Student-t distribution has variance $\nu/(\nu-2)$.

   How does the result compare with what you expect from the confidence level being 0.95*? What is the average width of the intervals? Draw conclusions!*

3. *Play around with the parameters to confirm conclusions.*

1.6.3 Robust Estimation

An ϵ-*contaminated* normal distribution may be defined as

$$X = WY + (1-W)Z = \begin{cases} Y & \text{with probability } 1-\epsilon \\ Z & \text{with probability } \epsilon \end{cases}.$$

Here $Y \sim \mathrm{N}(0, \sigma_Y^2)$, $Z \sim F$, and $W \sim \text{Bernoulli}(1-\epsilon)$, with $\epsilon \in (0,1)$ being a small number. Further, F is another distribution than the $\mathrm{N}(0, \sigma_Y^2)$ distribution, that usually displays much wilder fluctuation (i.e., more extreme values) than does the normal distribution.

This contaminated distribution can be viewed as that some phenomena usually is what is observed, at the rate of $1-\epsilon$, but that some other phenomena is observed, at the rate ϵ. In practice, this can be caused by somebody occasionally making a faulty measurement, or a sloppy computer registration of a result.

As the contaminated distribution is not normal, it can be difficult to analyze. In addition, when using a model of this kind, one is usually interested in the non-contaminated random variable Y, rather than the contaminated variable X.

One common way to handle the contaminated data, is to *remove outliers*. Notice that this is correct, if one is only interested in Y, but might be erroneous if really interested in the contaminated distribution of X.

1. *Use the contaminated distribution with $\sigma_Y = 1$, F Cauchy distributed with location parameter 0, and $\epsilon = 0.05$ to estimate the expected value of X 1000 times with a sample mean of size 100.*

   For simulation of the Cauchy distribution, one can make use of the fact that a Cauchy distribution with location parameter 0 is the same thing as a Student-t distribution, with 1 degree of freedom.

2. *Order the results and register the value of results number 25 and 975.*[6] *Also, make histogram plots of the result.*

3. *Repeat the above tasks, but this time replacing the sample mean with the robust estimators, made up of the sample median, and of the α-*trimmed mean

$$\bar{X}_\alpha = \frac{X_{(k+1)} + \ldots + X_{(n-k)}}{n - 2k} \qquad \text{with} \qquad \alpha = \frac{k}{n},$$

   respectively, where and $X_{(.)}$ again is the ordered sample. Choose $\alpha = 0.1$.

   Again, order the results and register the value of results number 25 and 975. Plot histograms of the results. Conclusions?

[6] The range between these two values does in fact make up a *bootstrap* confidence interval for the expected value of X, with confidence level $\alpha = 0.95$: We will return to this later in the course.