Chapter 5

Simulation of Stochastic Processes[5]

5.1 Stochastic processes

A stochastic process is a mathematical model for a random development in time:

Definition 5.1. *Let $T \subseteq \mathbb{R}$ be a set and Ω a sample space of outcomes. A* stochastic process *with* parameter space T *is a function* $X : \Omega \times T \to \mathbb{R}$.

A stochastic process with parameter space T is a family $\{X(t)\}_{t\in T}$ of random variables. For each value of the parameter $t \in T$ is the process value $X(t) = X(\omega, t)$ a random variable.

The parameter t is called *time parameter*. We distinguish between *discrete time*, where usually $T = \mathbb{Z}$ or $T = \mathbb{N}$, and *continuous time*, where usually $T = \mathbb{R}$ or $T = \mathbb{R}^+$. (Of course, a process with continuous time need not be continuous!)

Just like the dependence of $\omega \in \Omega$ is seldom indicated in the notation, for a random variable $\xi = \xi(\omega)$, it is customary to write $X(t)$ instead of $X(\omega, t)$, for a stochastic process. But just as ξ depends on the outcome ω, and is random, so does $X(t)$.

A plot of a stochastic process $X(\omega, t)$ as a function of $t \in T$, for a fixed outcome $\omega \in \Omega$, is called a *sample path* of the process: When a stochastic process is plotted, it is a (hopefully "typical") sample path that is depicted.

Example 5.1. A basic very important stochastic process in discrete time is *discrete white noise* $\{e(t)\}_{t\in\mathbb{Z}}$, where $\ldots, e(-1), e(0), e(1), \ldots$ are random variables such that

$$\mathbf{E}\{e(t)\} = 0, \quad \mathbf{Var}\{e(t)\} = \sigma^2 \quad \text{and} \quad \mathbf{Cov}\{e(s), e(t)\} = 0 \;\text{ for } s \neq t.$$

The next example concerns a stochastic process that is sort of a counter part to the discrete white noise that features in Example 5.1.

[5]For an economical ride through this chapter, omit Section 5.2-5.4 and Theorems 5.2-5.6.

Example 5.2. Consider the process $X(t) = \xi$ for $t \in \mathbb{Z}$, where ξ is a single random variable. A sample path $\{(t, X(t)) \in \mathbb{R}^2 : t \in \mathbb{Z}\}$, of this process is just the horisontal sequence of equidistant points $\{(t, \xi)) \in \mathbb{R}^2 : t \in \mathbb{Z}\}$. In other words, the process values are completely dependent of each other: If you know one process value, you all of them!

Figure 5.1. *Discrete Gaussian white noise with variance $\sigma^2 = 1$.*

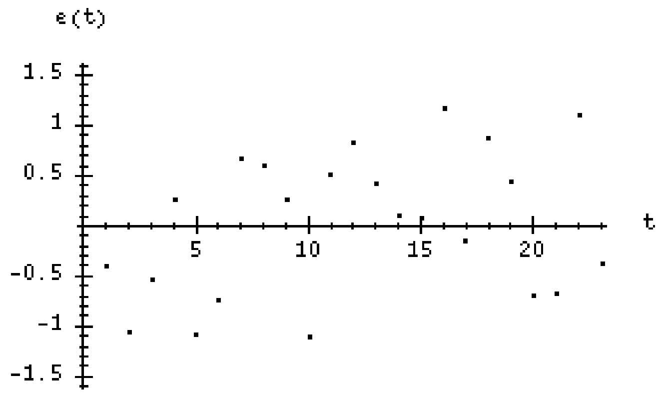
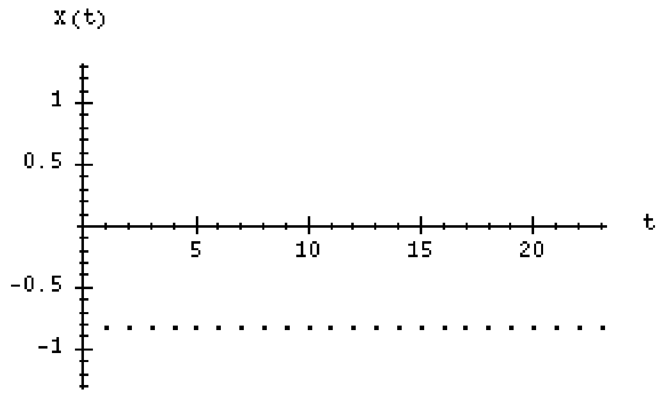


Figure 5.2. *The process in Example 5.2 with ξ $N(0,1)$ distributed.*



If the random variables $\ldots, e(-1), e(0), e(1), \ldots$ in Example 5.1 have the same distribution as the random variable ξ in Example 5.2, then the marginal distribution functions coincide, $F_{e(t)}(x) = F_{X(t)}(x)$ for all $t \in \mathbb{Z}$. Despite of this, the processes $\{e(t)\}_{t\in\mathbb{Z}}$ and $\{X(t)\}_{t\in\mathbb{Z}}$ are more or less as different as a pair of processes can be, as is illustrated by Figures 5.1 and 5.2: The first one with completely uncorrelated process values, and the second one with completely dependent process values.

Usually, in both theory and applications, process values $X(s)$ and $X(t)$, at different times s and t, depend more of one another than they do in Example 5.1, but less than in Example 5.2. The analysis of stochastic processes is much about the study of that dependence.

The analysis of the dependence structure of a stochastic process requires, in priciple, a complete knowledge of the *finite dimensional distributions* of the process, given by

$$F_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n)=\mathbf{P}\{X(t_1)\le x_1,\ldots,X(t_n)\le x_n\}\quad\text{for } t_1,\ldots,t_n\in T,$$

for all $n\in\mathbb{N}$. However, in practice these distributions can seldom be calculated. Instead of an exact analyzis, one must therefore often feel satisfied with approximations. Computer simulation of processes are often important aids, to find such approximations.

5.2 The Poisson process and other Lévy processes

Poisson processes feature already in basic course in statistics:

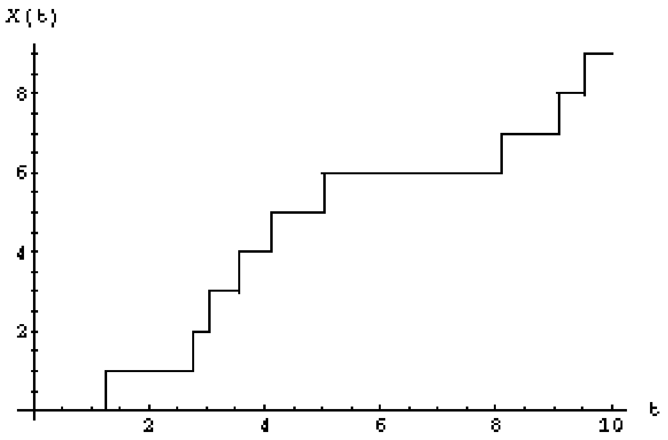
Definition 5.2. *Let* $\xi_1,\xi_2,\ldots$ *be independent* $\exp(\lambda)$ *distributed random variables. A* Poisson process $\{X(t)\}_{t\ge 0}$ *with* intensity λ *is given by*

$$X(t)=\begin{cases}0 & \textit{for} \quad 0\le t<\xi_1,\\ 1 & \textit{for} \quad \xi_1\le t<\xi_1+\xi_2,\\ \vdots & \qquad\qquad \vdots\\ n & \textit{for} \quad \xi_1+\ldots+\xi_n\le t<\xi_1+\ldots+\xi_n+\xi_{n+1},\\ \vdots & \qquad\qquad \vdots\end{cases}.$$

It is easy to simulate a Poisson process, because the definition is constructive, i.e., a recipy is given for how to create the process.

Example 5.3. The following Mathematica program simulates a sample path $\{X(t)\}_{t\in[0,10]}$ of a Poisson process with intensity $\lambda=1$, by mimicing Definition 5.2:

```
In[1]:= <<Statistics`ContinuousDistributions`
In[2]:= xi=N[Table[Random[ExponentialDistribution[1]],{i,1,50}]];
In[3]:= steg[t_]:=If[t>=0,1,0]
In[4]:= X[t_]:=Sum[steg[t-Sum[xi[[i]],{i,1,k}]],{k,1,50}]
In[5]:= Plot[X[t],{t,0,10},AxesLabel->{"t","X(t)"},PlotPoints->99]
```



Lévy processes are among the few most important classes of stochastic processes:

Definition 5.3. *A process* $\{X(t)\}_{t\geq 0}$ *has* independent increments *if, for each choice of* $n \in \mathbb{N}$ *and* $0 \leq t_0 < t_1 < \ldots < t_n$*, the random variables*

$$X(t_n) - X(_{n-1}), \ldots, X(t_2) - X(t_1), X(t_1) - X(t_0) \quad \textit{are independent.}$$

A process $\{X(t)\}_{t\geq 0}$ *has* stationary increments *if, for every* $h \geq 0$,

$$X(t+h) - X(t) =_{distribution} X(h) - X(0).$$

A process $\{X(t)\}_{t\geq 0}$ *is a* Lévy process *if increments are independent and stationary.*

In order to simulate, for example, the process values $\{X(\frac{1}{n})\}_{i=1}^{n}$ of a Lévy process (where $\frac{1}{n}$ typically is "small"), simulate n independent random variables $\{\zeta_i\}_{i=1}^{n}$ that are distributed as $X(\frac{1}{n})$. Then $\{\sum_{j=1}^{i} \zeta_j\}_{i=1}^{n}$ has the same joint distribution as $\{X(\frac{1}{n})\}_{i=1}^{n}$.

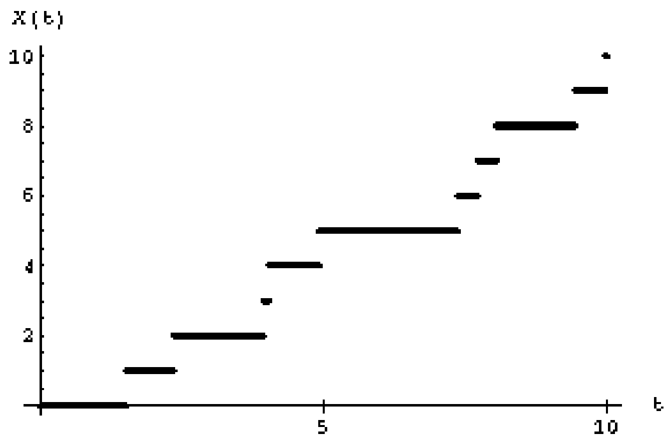
Theorem 5.1. *A Poisson process* $\{X(t)\}_{t\geq 0}$ *is a Lévy process with* $X(0) = 0$ *and*

$$X(t+h) - X(t) \;\; Po(\lambda h) \;\; \textit{distributed for} \;\; t, h \geq 0.$$

In Example 5.3, a Poisson process is simulated directly, by use of Definition 5.2. Since Poisson processes are Lévy processes, they can also be simulated according to the general recipy for Lévy processes, provided above.

Example 5.3. (Continued) The following Mathematica program simulates a sample path $\{X(t)\}_{t\in[0,10]}$ of a Poisson process with intensity $\lambda = 1$, by sampling the process 100 times per unit of time, so that increments are Po(0.01) distributed.

```
In[1]:= <<Statistics‘DiscreteDistributions‘
In[2]:= incr=N[Table[Random[PoissonDistribution[0.01]],
        {i,1,1000}]];
In[3]:= X=Table[Sum[incr[[i]],{i,1,k}],{k,1,1000}];
In[4]:= ListPlot[X,Ticks->{{{500,"5",0.02},{1000,"10",0.02}},
          Automatic},AxesLabel->{"t","X(t)"}]
```



5.3 Moment functions

A stochastic process $X(t)$ cannot be specified by its univariate marginal distribution only, as they do not give information of the dependence structure of the process (see Section 5.1). Moreover, although $X(t)$ can be specified by its finite dimensional distributions, these can seldom be expressed analytically in practice, in closed form.

As a middle way, between specifying the univariate marginals only, which say to little, and specifying the finite dimensional distributions, which usually cannot be done, one often describe processes by their *moment functions.* This description is quite incomplete, and does not determine processes uniquely, at all. However, knowledge of moment functions is enough for many important applications as, for example, often is the case in applications of filtering theory.

Definition 5.4. *A process* $\{X(t)\}_{t\in T}$ *has* expectation function $m_X : T \to \mathbb{R}$ *given by* $m_X(t) = \mathbf{E}\{X(t)\}$.

A process $\{X(t)\}_{t\in T}$ *has* covariance function $r_X : T \times T \to \mathbb{R}$ *given by* $r_X(s,t) = \mathbf{Cov}\{X(s), X(t)\}$.

The expectation function $m_X(t)$ gives information about the one-dimensional marginal distributions $F_{X(t)}(x) = \mathbf{P}\{X(t) \leq x\}$, $t \in T$, of the process $X(t)$. Notice that $m_X(t)$ is the center of gravity of the graph of the probability density function $f_{X(t)}(x)$ of $X(t)$.

The covariance function $r_X(s,t)$ gives information about the two-dimensional distributions $F_{X(s),X(t)}(x,y) = \mathbf{P}\{X(s) \leq x, X(t) \leq y\}$, $s,t \in T$, of $X(t)$. It gives a measurement of the degree of linear dependence between the random variables $X(s)$ and $X(t)$. Notice that $r_X(t,t) = \mathbf{Var}\{X(t)\}$.

Example 5.3. (Continued) For a Poisson process $X(t)$ one has, by Theorem 5.1,

$$m_X(t) = \mathbf{E}\{X(t)\} = \mathbf{E}\{\mathrm{Po}(\lambda t)\} = \lambda t.$$

Furthermore, $r_X(s,t) = \lambda \min\{s,t\}$, because independence of increments give

$$\begin{aligned} r_X(s,t) = \mathbf{Cov}\{X(s), X(t)\} &= \mathbf{Cov}\{X(s), X(t) - X(s)\} + \mathbf{Cov}\{X(s), X(s)\} \\ &= 0 + \mathbf{Var}\{X(s)\} \\ &= \mathbf{Var}\{\mathrm{Po}(\lambda s)\} = \lambda s \quad \text{for } s \leq t. \end{aligned}$$

5.4 Stationarity

In many branches of science, some characteristic of the type time invariance or steady state is of importance. Of course, such characteristics often simplify analysis.

A stationary stochastic processes has finite dimensional distributions that are *invariant under translations of time*:

Definition 5.5. *A process* $\{X(t)\}_{t\in T}$ *is* stationary *if for every selection of* $n \in \mathbb{N}$, $h \in \mathbb{R}$ *and* $t_1, \ldots, t_n, t_1 + h, \ldots, t_n + h \in T$, *one has*

$$(X(t_1), \ldots, X(t_n)) =_{distribution} (X(t_1 + h), \ldots, X(t_n + h)).$$

Example 5.1. (Continued) If the process values $\ldots, e(-1), e(0), e(1), \ldots$ for discrete white noise are independent and indentically distributed, then $\{e(t)\}_{t\in\mathbb{Z}}$ is a stationary process, because

$$\begin{aligned}\mathbf{P}\{e(t_1{+}h)\le x_1,\ldots,e(t_n{+}h)\le x_n\} &= \mathbf{P}\{e(t_1{+}h)\le x_1\}\times\ldots\times\mathbf{P}\{e(t_n{+}h)\le x_n\}\\ &= \mathbf{P}\{e(t_1)\le x_1\}\times\ldots\times\mathbf{P}\{e(t_n)\le x_n\}\\ &= \mathbf{P}\{e(t_1)\le x_1,\ldots,e(t_n)\le x_n\}.\end{aligned}$$

Example 5.2. (Continued) The process $X(t)=\xi$ for $t\in\mathbb{Z}$, where ξ is one single random variable, is stationary, because

$$\mathbf{P}\{X(t_1+h)\le x_1,\ldots,X(t_n+h)\le x_n\} = \mathbf{P}\{\xi\le \min[x_1,\ldots,x_n]\}$$

does not depend on the value of $h\in\mathbb{Z}$.

The following weaker form of stationarity is convenient in many applications:

Definition 5.6. *A process* $\{X(t)\}_{t\in T}$ *is* weakly stationary *if the expectation and covariance functions*

$$m_X(t) \quad \textit{and} \quad r_X(t,t+\tau) \quad \textit{do not depend on} \;\; t\in T.$$

Example 5.1. (Continued) Discrete white noise $\{e(t)\}_{t\in\mathbb{Z}}$ is weakly stationary, as $m_e(t)=\mathbf{E}\{e(t)\}=0$ and

$$r_e(t,t+\tau)=\mathbf{Cov}\{e(t),e(t+\tau)\}=\begin{cases}\sigma^2 & \text{if } \;\tau=0\\ 0 & \text{if }\;\tau\neq 0\end{cases}$$

do not depend on t.

Compared with the concept of stationarity, for a weakly stationary process, invariance under translation of the finite dimensional distribution

$$(X(t_1+h),\ldots,X(t_n+h)) =_{\text{distribution}} (X(t_1),\ldots,X(t_n)),$$

has been weakened to invariance under translation of the expectation and covariance functions

$$\mathbf{E}\{X(t_1+h)\}=\mathbf{E}\{X(t_1)\} \quad\text{and}\quad \mathbf{Cov}\{X(t_1+h),X(t_2+h)\}=\mathbf{Cov}\{X(t_1),X(t_2)\} \tag{5.1}$$

[as $m_X(t)$ does not depend on t, and $r_X(s,t)$ depends only on the difference $t-s$].

The concept of the weak stationarity only depends on finite dimensional distributions up to the second order. Further, weak stationarity does not require these distributions themselves to be translation invariant, but only that their expectaions and covariances are.

Disregarding possible problem with existence of moment, we have directly from the above discussion the following result:

Theorem 5.2. *A stationary process is weakly stationary.*

For a weakly stationary process $X(t)$, one usually writes m_X instead of $m_X(t)$, for the expectation function, and $r_X(\tau) = \mathbf{Cov}\{X(t), X(t+\tau)\}$ instead of $r_X(t, t+\tau)$ (which does not depend on t), for the covariance function.

5.5 Shot noise

Let signal packages with a certain form $g(t)$ arrive with $\exp(\lambda)$ distributed inter arrival times to a system. Then the summarized total signal in the system is *shot noise*:

Definition 5.7. *Let* $\xi_1, \xi_2, \ldots, \eta_1, \eta_2, \ldots$ *be independent* $\exp(\lambda)$ *distributed random variables and* $g : \mathbb{R} \to \mathbb{R}$ *a function satisfying* $\int_{-\infty}^{\infty} |g(x)|dx < \infty$. *The corresponding* shot noise process *is defined by*

$$X(t) = \sum_{k=1}^{\infty} g\Big(t - \sum_{\ell=1}^{k} \xi_\ell\Big) + \sum_{k=1}^{\infty} g\Big(t + \sum_{\ell=1}^{k} \eta_\ell\Big) \quad \textit{for} \quad t \in \mathbb{R}.$$

Since the definition of shot noise is constructive, it is straightforward to simulate.

An alternative way to describe shot noise, is by the relation

$$X(t) = \int_{-\infty}^{\infty} g(t-s)dY(s) + \int_{-\infty}^{\infty} g(t+s)dZ(s),$$

where $\{Y(s)\}_{s\geq 0}$ and $\{Z(s)\}_{s\geq 0}$ are independent Poisson processes with intensities λ. Using this description, it is straightforward to prove the following result:

Theorem 5.3. *A shot noise process* $\{X(t)\}_{t\in\mathbb{R}}$ *is weakly stationary with expectation and covariance functions*

$$m_X = \lambda \int_{-\infty}^{\infty} g(s)ds \quad \textit{and} \quad r_X(\tau) = \lambda \int_{-\infty}^{\infty} g(s)g(\tau+s)dx.$$

5.6 Gaussian Processes

Argubly, *Gaussian processes*, or *normal processes*, are the most important stochastic processes. They are recognized by linear combination of their process values are normal distributed random variables, or, equivalently, by that their finmite dimensional distribution are multivariate normal distributed. An important consequence of this is that outputs from linear filters with Gaussian input processes are again Gaussian.

Definition 5.8. *A process* $\{X(t)\}_{t\in\mathcal{T}}$ *is* Gaussian *if* $\sum_{k=1}^{n} a_k X(t_k)$ *is normal distributed for each choice of* $n \in \mathbb{N}$, $t_1, \ldots, t_n \in T$ *and* $a_1, \ldots, a_n \in \mathbb{R}$.

The definition of a Gaussian process is not constructive, and does not give a recipy according to which Gaussian processes can be simulated.

In order to specify a Gaussian process, it is enough to specify its expectation and covariance functions:

Theorem 5.4. *If the process $\{X(t)\}_{t\in T}$ is Gaussian, then the finite dimensional distributions are determined by the expectation function $m_X(t)$ together with the covariance function $r_X(s,t)$.*

Proof. The distribution of the Gaussian vector $(X(t_1),\ldots,X(t_n))$ is determined by its characteristic function

$$\phi_{X(t_1),\ldots,X(t_n)}(s_1,\ldots,s_n) = \mathbf{E}\{\mathrm{e}^{i(s_1X(t_1)+\ldots+s_nX(t_n))}\},$$

which in turn coincides with the characteristic function of the random variable $s_1X(t_1)+\ldots+s_nX(t_n)$, calculated in the point 1,

$$\phi_{s_1X(t_1)+\ldots+s_nX(t_n)}(1) = \mathbf{E}\{\mathrm{e}^{i\cdot 1\cdot(s_1X(t_1)+\ldots+s_nX(t_n))}\}.$$

Now, according to Definition 5.8, $\sum_{i=1}^n s_iX(t_i)$ is normal distributed, and thus determined by the expectation and variance

$$\begin{cases} \mathbf{E}\{\sum_{i=1}^n s_iX(t_i)\} = \sum\limits_{k=1}^n s_k\mathbf{E}\{X(t_k)\} = \sum\limits_{k=1}^n s_k m_X(t_k) \\ \mathbf{Var}\{\sum_{i=1}^n s_iX(t_i)\} = \sum\limits_{k=1}^n\sum\limits_{\ell=1}^n s_ks_\ell\mathbf{Cov}\{X(t_k),X(t_\ell)\} = \sum\limits_{k=1}^n\sum\limits_{\ell=1}^n s_ks_\ell r_X(t_k,t_\ell) \end{cases}$$ □

Gaussian random variables are independent if and only if they are uncorrelated:

Theorem 5.5. *The components of a Gaussian random vector are independent if and only if they are uncorrelated.*

Proof. The implication to the right is elementary. For the other implication, notice that dependent Gaussian random variables cannot be uncorrelation, because then they have the same correlations as independent Gaussian random variables, and thus have the same finite dimensional distributions, by Theorem 5.4, which would make them independent. □

Theorem 5.6. *A Gaussian process is stationary if and only if it is weakly stationary.*

Proof. The implication to the right follows from Theorem 5.2. On the other hand, by Theorem 5.4, $X(t)$ is stationary, i.e., the process $\{X(t+h)\}_{t\in T}$ has the same finite dimensional distributions as $\{X(t)\}_{t\in T}$, for every h, if $X(t+h)$ and has the same expectation and covariance function as do $X(t)$. That this holds, in turn, follows from weak stationarity, according to (5.1). □

Not every function $r(s,t)$ is a covariance function, in the same way as not every matrix is a covariance matrix. However, we know from Example 5.3 that the function $r(s,t) = \min(s,t)$ is really a covariance function.

Definition 5.9. *A Gaussian process* $\{W(t)\}_{t\in\mathbb{R}}$ *is a* (standard) Wiener process *if it has expectation and covariance functions given by*

$$m_W(t)=0 \quad \textit{and} \quad r_W(s,t)=\begin{cases}\min(|s|,|t|) & \textit{for} \quad s,t\le 0\,,\\ 0 & \textit{for} \quad \min\{s,t\}\le 0\le \max\{s,t\},\\ \min(s,t) & \textit{for} \quad 0\le s,t.\end{cases}$$

The Wiener process is a Lévy process:

Theorem 5.7. *A Wiener process* $\{W(t)\}_{t\in\mathbb{R}}$ *is a Lévy process with* $W(0)=0$ *and*

$$W(t+h)-W(t)\ \ N(0,h)\ \textit{distributed for}\ \ t\in\mathbb{R}\ \ \textit{and}\ \ h\ge 0.$$

Proof. To show that the increments of $W(t)$ are independent, by Theorem 5.5, it is enough to show that they are uncorrelated. This in turn follows, because

$$\begin{aligned}&\mathbf{Cov}\{W(t_i)-W(t_{i-1}),W(t_j)-W(t_{j-1})\}\\ &\qquad=\min\{t_i,t_j\}-\min\{t_i,t_{j-1}\}-\min\{t_{i-1},t_j\}+\min\{t_{i-1},t_{j-1}\}\\ &\qquad=t_i-t_i-t_{i-1}+t_{i-1}\\ &\qquad=0\end{aligned}$$

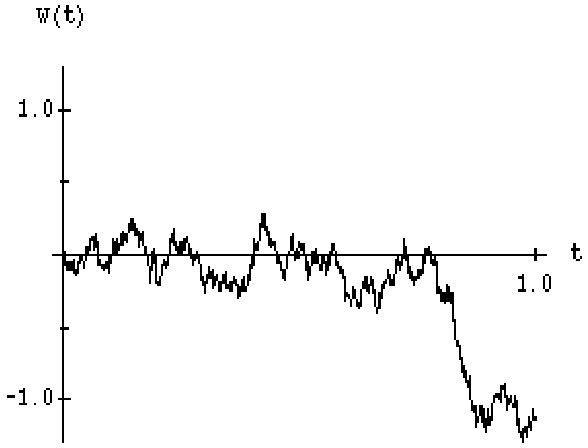
for $0\le t_{i-1}<t_i\le t_{j-1}<t_j$. The case when some of the times $t_{i-1}<t_i\le t_{j-1}<t_j$ are negative is similarly dealt with. Furthermore, $W(t+h)-W(t)$ is $N(0,h)$ distributed, since $\mathbf{E}\{W(t+h)-W(t)\}=0$, and

$$\begin{aligned}\mathbf{Var}\{W(t+h)-W(t)\}&=\mathbf{Var}\{W(t+h)\}-2\mathbf{Cov}\{W(t+h),W(t)\}+\mathbf{Var}\{W(t)\}\\ &=t+h-2t+t\\ &=h\quad\text{for}\ \ t\ge 0,\end{aligned}$$

where the case when $t\le 0$ is similarly dealt with. Finally, we have $W(0)=0$ since $\mathbf{E}\{W(0)\}=m_W(0)=0$ and $\mathbf{Var}\{W(0)\}=r_W(0,0)=0$. □

Example 5.2. The following Mathematica program simulates a sample path $\{W(t)\}_{t\in[0,1]}$ for a standard Wiener process, by sampling the process 1000 times per unit of time, so that increments are N(0.001) distributed.

```
In[1]:= <<Statistics`ContinuousDistributions`
In[2]:= W=Table[Random[NormalDistribution[0,Sqrt[0.001]]],
            {i,1,1000}]
In[3]:= WProc=Table[Sum[W[[i]],{i,1,j}],{j,1,1000}]
In[4]:= ListPlot[WProc,AxesLabel->{"t","W(t)"},PlotJoined
            ->True,PlotRange->{-1.3,1.3},Ticks->{{{1000,"1.0"}},
            {{-1.0,"-1.0"},{-0.5,},{0.5,},{1.0,"1.0"}}}]
```



5.7 Simulation of stationary Gaussian processes

Let $\{W(t)\}_{t\in\mathbb{R}}$ be a Wiener-process and $f:\mathbb{R}\to\mathbb{R}$ a function such that $\int_{-\infty}^{\infty} f(s)^2 ds < \infty$. It is not too hard to show that the following stochastic integral is well defined.

$$\int_{-\infty}^{\infty} f(s)dW(s) = \lim_{n\to\infty} \sum_{k=-s(n)}^{s(n)} f\Big(\frac{k}{n}\Big)\Big(W\Big(\frac{k+1}{n}\Big) - W\Big(\frac{k}{n}\Big)\Big).$$

Here the summation limits $\pm s(n)$ should satisfy $\lim_{n\to\infty} s(n)/n = \infty$.

Theorem 5.8. *The stochastic process*

$$X(t) = \int_{-\infty}^{\infty} f(t+s)dW(s) \quad \textit{for } t \in \mathbb{R}$$

is Gaussian and stationary, with expectation and covariance functions

$$m_X(t) = 0 \quad \textit{and} \quad r_X(\tau) = \int_{-\infty}^{\infty} f(s)f(\tau+s)ds.$$

Proof. Since $X(t)$ is a (limit of) linear combinations of normal distributed variables, $X(t)$ is normal distributed, by Definition 5.8. In the same fashion, it is seen that linear combinations of values of $X(t)$ are normal distributed. Hence $X(t)$ is a Gaussian process.

Since the Wiener process has zero expectation, it is quite obvious that $m_X(t) = 0$.

Using that

$$\mathbf{Cov}\Big\{W\Big(\frac{k+1}{n}\Big) - W\Big(\frac{k}{n}\Big), W\Big(\frac{\ell+1}{n}\Big) - W\Big(\frac{\ell}{n}\Big)\Big\} = \begin{cases} 1/n & \text{if } k=\ell \\ 0 & \text{if } k\neq\ell \end{cases},$$

it follows that the also the covariance function is as claimed

$$
\begin{aligned}
& r_X(t,t+\tau) \\
\leftarrow\ & \mathbf{Cov}\Big\{\sum_{k=-s(n)}^{s(n)} f\Big(t+\frac{k}{n}\Big)\Big(W\Big(\frac{k+1}{n}\Big)-W\Big(\frac{k}{n}\Big)\Big),\sum_{\ell=-s(n)}^{s(n)} f\Big(t+\tau+\frac{\ell}{n}\Big)\Big(W\Big(\frac{\ell+1}{n}\Big)-W\Big(\frac{\ell}{n}\Big)\Big)\Big\} \\
=\ & \sum_{k=-s(n)}^{s(n)} f\Big(t+\frac{k}{n}\Big)f\Big(t+\tau+\frac{k}{n}\Big)\frac{1}{n} \\
\rightarrow\ & \int_{-\infty}^{\infty} f(t+s)f(t+\tau+s)ds = \int_{-\infty}^{\infty} f(\hat{s})f(\tau+\hat{s})d\hat{s}.
\end{aligned}
$$

Thus $X(t)$ is weakly stationary, and therefore also stationary, according to Theorem 5.6. □

To simulate a zero-mean stationary Gaussian process $X(t)$, with covariance function $r(\tau)$, first pick a sufficiently large n and $s(n)$. Then approximate $X(t)$ by

$$X(t) \approx \sum_{k=-s(n)}^{s(n)} f\Big(t+\frac{k}{n}\Big)\Big(W\Big(\frac{k+1}{n}\Big)-W\Big(\frac{k}{n}\Big)\Big),$$

where the function f is given by

$$f(t) = (\mathcal{F}^{-1}\sqrt{\mathcal{F}r})(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\int_{-\infty}^{\infty} r(\tau)e^{i\nu\tau}d\tau}\, e^{-i\nu t}d\nu.$$

This is so, because then the covariance function $(f*f(-\cdot))(\tau) = \int_{-\infty}^{\infty} f(s)f(\tau+s)ds$ of the resulting Gaussian process (see Theorem 5.8) gets the right Fourier transform

$$(\mathcal{F}(f*f(-\cdot)))(\nu) = |(\mathcal{F}f)(\nu)|^2 = ((\sqrt{\mathcal{F}r})(\nu))^2 = (\mathcal{F}r)(\nu).$$

5.8 Laboration

1. Let $\{X(t)\}_{t\in\mathbb{R}}$ be a shot noise process with $\lambda = 1$ and

$$g(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad 0 \le t \le \frac{1}{2} \\ 0 & \text{for} \quad \frac{1}{2} < t \end{cases}.$$

The process $X(t)$ can be thought of as modelling the number of vehicles on a bridge at time t, when inter arrival times for vehicles are exp(1) distributed, and it takes vehicles $\frac{1}{2}$ a unit of time to pass the bridge.

a) *Simulate a sample path of $\{X(t)\}_{t\in[0,10]}$ and show the result in a plot.*

b) For the probability $p = \mathbf{P}\{\max_{t\in[0,10]} X(t) \ge 3\}$ we have

$$p = \mathbf{E}\{\zeta\} \quad \text{where} \quad \zeta = \begin{cases} 1 & \text{if} \quad \max_{t\in[0,10]} X(t) \ge 3, \\ 0 & \text{if} \quad \max_{t\in[0,10]} X(t) < 3. \end{cases}$$

Make a confidence interval on the 99% level for p based on 10000 *simulations of the random variable ζ.*

It can be of considerable help to notice that, writing

$$\mathbb{M} \equiv \{-(\eta_1+\ldots+\eta_{K_2}),\ldots,-(\eta_1+\eta_2),-\eta_1,\xi_1,\xi_1+\xi_2,\ldots,\xi_1+\ldots+\xi_{K_1}\}, \quad (5.2)$$

with the numbers K_1 and K_2 given below, we have

$$X(t)=\sum_{k=1}^{\infty} g\Big(t-\sum_{\ell=1}^{k}\xi_\ell\Big)+\sum_{k=1}^{\infty} g\Big(t+\sum_{\ell=1}^{k}\eta_\ell\Big)\geq 3 \quad \text{for some } t\in[0,10],$$

if and only if there exist numbers $m_1,m_2,m_3\in\mathbb{M}$ such that

$$-\frac{1}{2}\leq m_1<m_2<m_3\leq 10 \quad \text{and} \quad m_3-m_1\leq\frac{1}{2}. \quad (5.3)$$

Because of the form of the function $g(t)$, to simulate $\{X(t)\}_{t\in[0,10]}$, we need to consider $\xi_1,\ldots,\xi_{K_1}$, where K_1 is the largest integer such that $\xi_1+\ldots+\xi_{K_1}\leq 10$. In the same way, we need to consider $\eta_1,\ldots,\eta_{K_2}$, where K_2 is the largest number such that $\eta_1+\ldots+\eta_{K_2}\leq 0.5$ (often $K_2=0$). We can then compute

$$X(t)=\sum_{k=1}^{K_1} g\Big(t-\sum_{j=1}^{k}\xi_j\Big)+\sum_{k=1}^{K_2} g\Big(t+\sum_{j=1}^{k}\eta_j\Big) \quad \text{for } t\in[0,10]. \quad (5.4)$$

To complete Task 1 a, we do the following:

1) Simulate the exp(1) disributed random variables $\xi_1,\ldots,\xi_{K_1},\eta_1,\ldots,\eta_{K_2}$.

2) Calculate $X(t)$ from equation (5.4).

3) Plot $X(t)$.

In Task 1 b a loop is used, that do the following $n=10000$ times (select a smaller n when the program is tested):

1) As above.

2) Check if the condition (5.3) holds for some choice of m_1,m_2,m_3 in the set $\mathbb{M}$ given by (5.2). If this is the case, then $\max_{t\in[0,10]}X(t)\geq 3$, so that $\zeta=1$, while otherwise, this is not the case, so that $\zeta=0$.

The following S^+-program might be useful:

```
translvekt <- null; xisum <- -log(runif); etasum <- log(runif)
while (xisum<=10) {translvekt<-c(translvekt,xisum);
   xisum <- xisum - log(runif)}
while (etasum>=-0.5) {translvekt <- c(etasum,translvekt);
   etasum <- etasum + log(runif)}
X <- function(translvekt,t){n <- length(translvekt); X <- 0;
   for (i in 1:n) {X <- X + g(t-translvekt[n])};
   X}
```

2. Consider the stationary Gaussian process

$$X(t) = \int_{-\infty}^{\infty} f(t+s)dW(s) \quad \text{for } t \in \mathbb{R},$$

where $\{W(s)\}_{s\in\mathbb{R}}$ is a Wiener process and

$$f(t) = \begin{cases} 1-t^2 & \text{for } |t| \le 1 \\ 0 & \text{for } |t| > 1 \end{cases}.$$

Simulate a sample path of $\{X(t)\}_{t\in[0,10]}$ *and show the result with a plot.*