Chapter 2

Decision Theory

2.1 Introduction

Everyday one faces problems which call for decisions.

> **Example 2.1.** At a summer evening, a person makes plans for the next day, and chooses between going to the beach or to go shopping. If next day is sunny, the beach is the best choice, but if the weather is less good, it is best to go shopping. How should the person decide what plans to make (if really having to plan ahead, already the evening before)?

The difficulty in the above example is that the weather for tomorrow cannot be known with complete certainty, so that it is a question of a *decision under uncertainty.* Decision theory makes science of the art of making such decisions.

2.2 Decision theory and utility

Let us consider a situation in which we can take different actions, denoting the set of all possible actions available to us by $\mathcal{A}$. In the example above this set is quite small, it consists of only two possibilities: $a_1 =$ "go to the beach" and $a_2 =$ "go shopping". However, observe that generally the number of actions to choose from need not be finite and the set $\mathcal{A}$ can even be continuous. Let us also assume that there is a true state of nature, which we have no knowledge of, but which will determine how much we gain or lose by choosing one action out of those in $\mathcal{A}$. Call this set of all possible "nature states" for Θ. In the example this set again contains only two values: $\theta_1 =$ "sunny tomorrow" and $\theta_2 =$ "rains tomorrow". Similarly to $\mathcal{A}$, in more complex situations this set may be infinite and continuous. Finally, assume that for each a and θ it is possible for us to determine a function $u(a, \theta)$ that describes our gain if we perform a given that θ is true. This function, called utility, is a *subjective* measure of gain that is constructed according to a persons' views and beliefs and can not be "right" or "wrong". It is simply something that we choose to use for this particular situation. For the example above it can look something like in Table 2.1.

		States of nature	
		θ_1 = "sunny"	θ_2 = "rainy"
Actions	a_1 = "beach"	10	-5
	a_2 = "shopping"	2	6

Table 2.1: The value of a utility function $u(a, \theta)$ for Example 1.

Given θ it easy to see which a we should choose. Of course, the whole point is that we must pick an a while not knowing θ, that is we have to make a decision under uncertainty. So how, exactly, do we do that?

The answer is, we have to make a guess about θ. While we do not know its exact value, we may have reasons to believe (for example, from past experience) that one state of nature is more likely than another. If it is sunny today, it is reasonable to conclude that it is more probable that it will be sunny tomorrow than that it will rain. That is, according to our belief, different values of θ can be true with different probabilities and so we can define a probability distribution on the space Θ. This will enable us to talk about the *expected* utility $U_\pi(a)$ that will describe how much we expect to gain by choosing a given a probability distribution $\pi(\theta)$ on all the possible states of nature. Formally, we have

$$U_\pi(a) = \mathbf{E}_\Theta[u(a, \theta)] = \begin{cases} \int_\Theta u(a, \theta)\pi(\theta)d\theta & \text{for a continuous state space} \\ \sum_\Theta u(a, \theta)\pi(\theta) & \text{for a discrete state space} \end{cases}$$

For our simple example, the probability distribution $\pi(\theta)$ may be $\{\pi(\theta_1), \pi(\theta_2\} = \{0.8, 0.2\}$ leading to expected utilities

$$\begin{cases} U_\pi(a_1) & = 10 * 0.8 + (-5) * 0.2 = 7 \\ U_\pi(a_2) & = 2 * 0.8 + 6 * 0.2 = 2.8 \end{cases}$$

Clearly, given these particular $u(a, \theta)$ and $\pi(\theta)$ action a_1 = "go to the beach" is to be preferred.

2.2.1 Utility of money

Let us say that you have an amount of money S at your disposal. Assuming that you don't spend it right away, you can think of two possible actions: a_1 = "store it away" and a_2 = "invest it in stocks". In the first case the amount S will stay the same tomorrow as it is today, while in the second case it may increase but it may also decrease. Which of the actions do you prefer? The answer depends on how much you value the possible monetary gain and loss, that is on your utility.

The space of the states of nature Θ in this case consists of all possible differences in the prices of the stock between tomorrow and today. That is, Θ is continuous. However, small changes should be far more likely than large ones. Assume, for the sake of simplicity, that we have reason to believe that the probability of a negative difference is the same as a positive one. Then it is natural to choose our prior distribution $\pi(\theta)$ to be symmetric with light tails, centered around 0. The Normal distribution fulfills the requirements, so we let $\pi(\theta)$, again for simplicity, be $N(0,1)$.

Next, we choose a utility $u(a_2,\theta)$. Consider three choices: $u_1(a_2,\theta) = 1 - e^{-\theta}$ and $u_2(a_2,\theta) = e^{\theta} - 1$ and $u_3(a_2,\theta) = \theta$. The utilities are plotted in Figure 2.1. The third utility, the straight line, represents neutral behaviour. In this case using it reflects the belief that for you the worth of money is proportional (or equal) to its amount. The other two, $u_1(a_2,\theta)$ and $u_2(a_2,\theta)$ describe the so called risk averse and risk seeking behaviours. A risk averse utility is concave and favors gaining a small amount of money with certainty. It will also put a large negative weight on loosing your capital. According to the risk seeking utility, on the other hand, you find only large gains to be useful and do not care too much if your original capital S is lost.

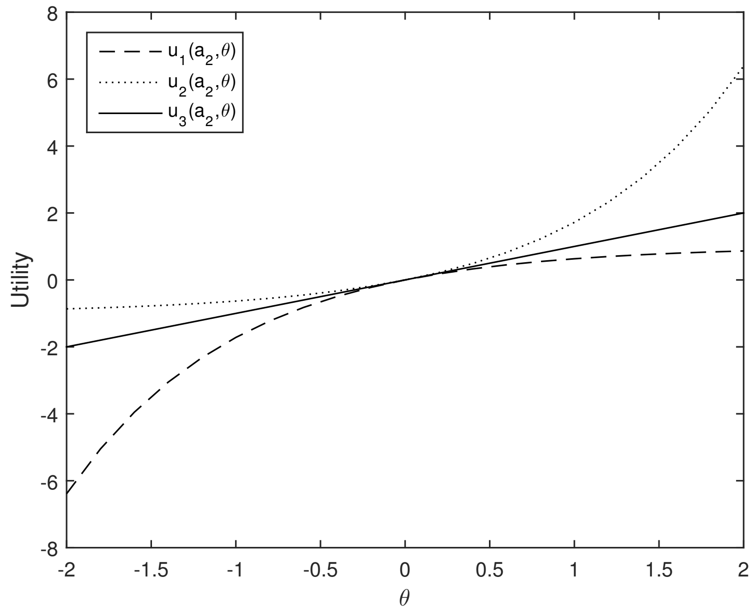

Figure 2.1: The risk averse (dashed), risk seeking (dotted) and risk neutral (solid) utilities.

The expected utilities in those three cases are going to be

$$U_\pi(a_2)_1 = \int_{-\infty}^{\infty} (1 - e^{-\theta})\pi(\theta)d\theta = -0.65$$
$$U_\pi(a_2)_2 = \int_{-\infty}^{\infty} (e^{\theta} - 1)\pi(\theta)d\theta = 0.65$$
$$U_\pi(a_2)_3 = \int_{-\infty}^{\infty} \theta\pi(\theta)d\theta = 0$$

Now we can answer the question of whether we should save our money or invest it. If you are a risk averse person, then your expected utility is going to be smaller than 0 (which is the expected utility of a_1) and choosing a_1 is preferable. If you are a risk seeker, then you should invest. Finally, if you are of the neutral disposition then a_1 and a_2 are equivalent in the sense that they have equal expected utilities.

2.3 Autocorrelation

Let $S(t)$ denote the price of a stock at time t and $\{S(t)\}_{t\in\mathbb{Z}}$ a collection of prices for different time points. This collection can be thought of as a non-stationary stochastic process, which is really hard to model. Instead, one tends to look at the standardized differences between the consecutive time points, that is *returns* and *log-returns*, defined as

$$\begin{aligned} X(t) &= (S(t) - S(t-1))/S(t-1) && \textit{returns} \\ X(t) &= \log(S(t)/S(t-1)) = \log(S(t)) - \log(S(t-1)) && \textit{log-returns} \end{aligned}$$

Based on historical evidence those standardized differences are believed to be independent from one day to another and identically normally distributed. The independence assumption means that no one should be able to predict the stock market tomorrow and be able to make money without risk.

The independence assumption, although generally believed to be valid, does not always need to be correct. In order to see whether it is plausible, we can try to model this possible dependence with an *autocorrelation function* defined as

$$r_X(s,t) = \frac{\mathbf{Cov}\{X(s), X(t)\}}{\sqrt{\mathbf{Var}\{X(s)\}\,\mathbf{Var}\{X(t)\}}}.$$

where $X(t)$ and $X(s)$ correspond to the values of a stochastic process at time points t and s. Under the assumption of *stationarity* (the distribution of $\{X(t)\}_{t=m}^{m+h}$ is independent of t for a fixed h), the ACF only depends on the time lag, h. This means that

$$r_X(h) = r_X(t, t+h) = \frac{\mathbf{Cov}\{X(t), X(t+h)\}}{\mathbf{Var}\{X(t)\}}.$$

Note that correlation between two random variables, and hence the ACF, only captures linear dependence. Remember that the correlation is always 0 for two independent random variables, but the opposite is not true.

Given a sequence of log-returns $X(0), X(1), \ldots, X(T)$, we can estimate the ACF by

$$\hat{r}_X(h) = \frac{(T+1)\sum_{i=0}^{T-h}(X(i) - \overline{X})(X(i+h) - \overline{X})}{(T-h+1)\sum_{i=0}^{T}(X(i) - \overline{X})^2},$$

where

$$\overline{X} = \frac{1}{T+1}\sum_{i=0}^{T} X(i).$$

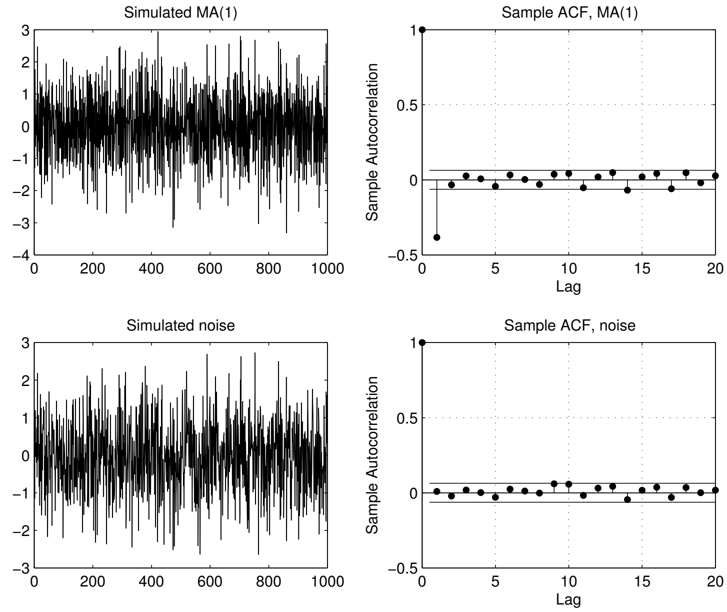

Figure 2.2: Top left plot: Simulated MA(1) with $a = 0.5$. Top right plot: Sample ACF for simulated MA(1). Straight line is the 95% confidence interval for independence. Bottom left plot: Simulated white noise. Bottom right plot: Sample ACF for simulated noise. Straight line is the 95% confidence interval for independence.

The function $\hat{r}_X(h)$ is known as the sample auto correlation function[1].

To investigate linear independence the question is how small the sample ACF must be to be interpreted as linear independence. It turns out that, under some assumptions, for independent data and with $n =$ number of time points,

$$\lim_{n\to\infty} \sqrt{n}\,\hat{r}_X(h) \sim N(0,1).$$

This can be used to construct a 95% confidence interval for independent data. Hence, for n sufficient large we assume linear independence if

$$|\hat{r}_X(h)| \leq 1.96/\sqrt{n}.$$

To investigate independence it is common to continue the analysis with estimation of the autocorrelation function for $f(X(0)), \ldots, f(X(T))$, for instance with $f(x) = |x|$. If one still has no linear dependence for the transformed data one assumes independence.

Example 2.2. A process $X(t) = \epsilon(t) - a\epsilon(t-1)$ where $\{\epsilon(t)\}_{t\in\mathbb{Z}}$ are zero-mean i.i.d. random variables is called a *moving average* process of degree 1 (MA(1)). Here, we have a dependence between $X(t)$ and $X(t+h)$ for $h = 1$, while we have independence for $h > 1$. In Figure 2.2 we

[1]This definition of the sample autocorrelation function differs slightly from the implementation of it in MATLAB (i.e. the function `autocorr`). This makes little difference in practice and you may use the MATLAB implementation.

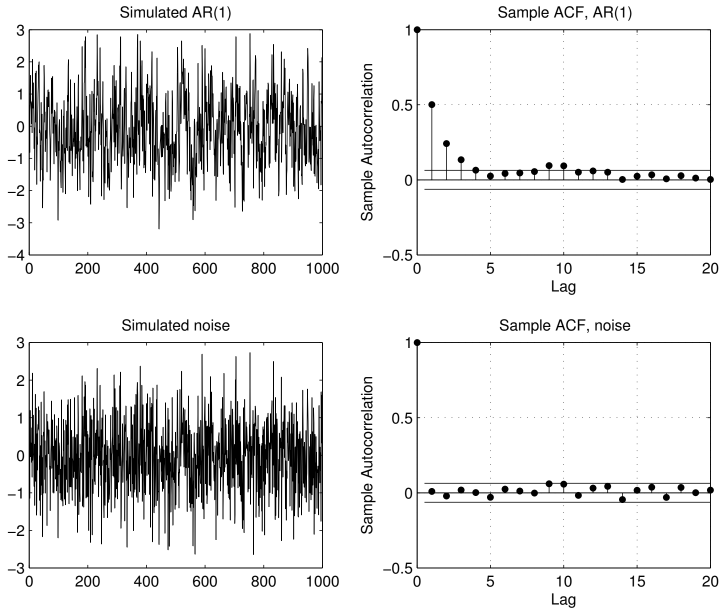

Figure 2.3: Top left plot: Simulated AR(1) with $a = 0.5$. Top right plot: Sample ACF for simulated AR(1). Straight line is the 95% confidence interval for independence. Bottom left plot: Simulated white noise. Bottom right plot: Sample ACF for simulated noise. Straight line is the 95% confidence interval for independence.

have simulated the process with noise modeled as $\epsilon(t) \sim N(0,1)$ and $a = 0.5$ and estimated the sample ACF. We have also plotted the noise and its sample ACF. Although the processes looks similar they have very different dependence structure.

Example 2.3. A process $X(t) = \epsilon(t) + aX(t-1)$ where $\{\epsilon(t)\}_{t\in\mathbb{Z}}$ are zero-mean i.i.d. random variables is called an *auto regressive* process of degree 1 (AR(1)). Here, we will have decreasing dependence between $X(t)$ and $X(t+h)$ for $h > 1$. In Figure 2.3 we have simulated the process with noise modelled as $\epsilon(t) \sim N(0,1)$, $a = 0.5$ and $X(0) = 0$ and estimated the sample ACF. We have also plotted the noise and its sample ACF. Although the processes looks similar they have very different dependence structure.

2.4 Computer assignment: Portfolio optimization

In order to get a pass on this lab, 7 out of 13 points are needed.

In this lab our objective is to create a portfolio (i.e. divide our capital S between n different stocks) in such a way that the difference between the value of the portfolio today and tomorrow is maximized, but taking the risk into account. That is, we would like to optimize it, with optimization performed over all possible divisions of S. It is those divisions that the action space $\mathcal{A}$ consists of, and we can describe each action a as a column vector of weights $w = [w_1, ..., w_n]^T$ such that $\sum_{i=1}^{n} w_i = 1$, where each weight corresponds to how much of our capital we invest in a particular stock.

The states of nature Θ are, similarly to the Section 2.2.1, all possible differences of the stock prices (i.e. returns or log-returns). However, now we do not have one single stock but, rather, n. As stated earlier, the (log)returns are considered to be time-independent and normally distributed. If this assumption is correct, we can model all the differences simultaneously with a joint multivariate normal distribution. The mean vector μ and the covariance matrix Σ can be estimated using historical data. Let $X_i(t)$ be the log-returns of stock i at time t. Observe also that the log-returns are assumed to be i.i.d. normal *in time* (that is, for instance, $X_1(t)$ is independent of $X_1(t-1)$), but it may or may not be true that $X_1(t)$ is independent of $X_2(t)$.

In this assignment you are given the prices for 7 different stocks available to play with, namely AstraZenica, Electrolux, Ericsson, Gambio, Nokia, Swedish Match and Svenska Handelsbanken. These are recorded in form of time series (that is $S(t)$ and not $X(t)$) from 2002-06-03 to 2006-06-01 and can be found in the file *stockdata.tsv*. The first column in the file is a date vector, with the other columns containing the 7 stock price developments.

Our portfolio is going to consist of the weighted sum of the log-returns, that is $\sum_{i=1}^{7} w_i X_i(t)$. Under the above multivariate normal assumption, the portfolio (i.e. the sum above) then becomes normally distributed with expected value $z = \mu^T w$ and variance $\sigma^2 = w^T \Sigma w$, where Σ is the covariance matrix.

We are going to examine a family of utility functions, where each utility has the form $u(a, \theta) = 1 - e^{-kw^T\theta}$, with k a positive. By denoting total (transformed) gain that the portfolio yields through $x = w^T\theta$, we can define the utility in terms of x rather than a and θ as $u(x) = 1 - e^{-kx}$. Then, the expected utility of the whole portfolio becomes

$$U_\pi(w) = \int_{-\infty}^{\infty} (1 - \mathrm{e}^{-kx}) \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-((x-z)^2/(2\sigma^2))}\, dx \tag{2.1}$$

and we face the problem of maximizing $U_\pi(w)$ under the constraints

$$w_1, \ldots, w_n \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} w_i = 1.$$

It can be shown that this problem is equivalent with the quadratic problem to maximize

$$\mu^T w - \frac{k}{2} w^T \Sigma w \tag{2.2}$$

under the same constraints. To show this is the task in Assignment 2. However, note that you are allowed to use the expression in equation 2.2 when maximizing the expected utility in Assignment 3 and 4 even if you choose not to complete Assignment 2.

2.4.1 Assignment 1, Data exploration (4p)

This first assignment aims at getting a feel of the dataset by ways of plots, histograms etc.

Assignment 1.1, Assumptions about the data

- Load the data and calculate log-returns for each stock. Use natural logarithms throughout this lab. Plot the log-returns for each stock against time. Also, display those in histograms and do qq-plots for each stock. (If the plots look similar, then you do not need to include all of them in the report.) Also perform a formal goodness of fit test for each stock. Do normal assumptions for the log-returns seem plausible?

- Estimate the autocorrelation function of the log-returns and also of the absolute values of the log-returns, for each stock. Plot it and interpret. Do the assumptions of independence between time points seem plausible?

- Estimate the mean and standard deviation for the log-returns of each stock. Also, estimate the covariance/correlation structure between the log-returns corresponding to different stocks. Do the they seem to be independent (do you expect them to be)?

- Do the data assumptions hold for the log returns from the seven stocks? What implications does this have on the analysis performed in this lab and the interpretation of the results? Don't forget to comment on this in the discussion of the report.

Assignment 1.2, Utility and expected utility

- Explore the utility function for different values of k. Present in a plot. Interpret in terms of risk aversion/seeking.

- Consider the log-returns for each stock at a time. Calculate the expected utility for a few different k (among those, $k = 1$). Which of the stocks is the most advantageous (this may be different for different k-values)? Comment on the result.

2.4.2 Assignment 2 (2p)

- Show that the problem of maximizing Equation 2.1 is, in fact, equivalent to the problem of maximizing Equation 2.2.

2.4.3 Assignment 3, Optimization with two stocks (3p)

In this part we are going to optimize the portfolio using only two stocks, namely Ericsson and Gambio.

- Take out the Ericsson and Gambio stocks from the data set. Estimate the mean vector and covariance matrix for the log-returns.
- Using those estimates and $k = 1$, calculate and plot the expected utility $U_\pi(w)$ for $w_1 \in [0, 1]$, $w_2 = 1 - w_1$. Look at the plot and answer the question: Approximately, for what w_1 do we have maximum expected utility?
- Find the optimum point more accurately with **fmincon** procedure. Repeat for several different k and report the optimal weights for each k. Do different stocks get different weight? If so, explain why one gets higher weight than another. Explain how and why the weights change when k is changed.

2.4.4 Assignment 4, Optimization with seven stocks (3p)

- Repeat the optimization in the previous task, for different values of k (including $k = 1$), using all seven stocks simultaneously. Report both the optimal weights in each case and the resulting expected utility. Also, for each chosen k, calculate and report $U_\pi(w)$ for the "naive" way of dividing money equally between the stocks.
- Compare and discuss the results above, and also consider the expected utility for putting everything into one stock (see Assignment 1).

2.4.5 Software and comments on the assignments

The lab will be performed in the Matlab environment.

- For this assignment, you may find the following functions in Matlab particularly useful: **load, hist, fmincon, quad, subplot**
- Reading the text about "Passing Extra Parameters" in the help pages can be of value while using, for example, **fmincon** and **quad**.
- If you have problems calculating the expected utility integral numerically with **quad**, namely you get it equal to zero all the time, set smaller integral limits. More precisely, those limits should be the places where the function inside the integral begins to level out, so plotting the said function first helps a lot.
- When you do the optimization with **fmincon** you may need to either change the tolerance or blow up your function values (by multiplying it with, say, 1000).

- Completing the second, theoretical, task, will give you a simpler expression of the expected utility. You are welcome to use it, instead of calculating the integral numerically, in Assignment 3 and 4.

- In order to get a pass on this lab, 7 out of 13 points are needed.

- Some help to the discussion in Assignment 3 and 4: When discussing the changes of weights with different choices of k, it could be of help to think about how the variance for a linear combination of (dependent) random variables is calculated and also to find out which values of the weights that give the lowest variance to the portfolio.