Stochastic Calculus Part I Fall 2009

Lecture 11 on Applications

1 Introduction

In this lecture we will give two applications of tools from stochastic calculus to the modelling of real-world financial data. The first application is to model the Stockholm Stock Exchange index by the stochastic exponential of Brownian motion (BM), that is, the Black-Scholes model. The second application is to model Nordpool spot market eletricity prices by means of an Ornstein-Uhlenbeck (OU) process, that is, the Langevin stochastic differential equation (SDE).

The issue whether we can establish a good fit of the above models to the data has to be investigated by statistical methodology. One motivation for the modelling, as often is the case in science, is that with a theoretical model we can use theory to calculate various properties of the model, which we can hope that they fit with the corresponding properties of the modelled real-world phenomena if the model is good enough.

2 Elements of diffusion theory

A *time homogeneous diffusion process* is the solution $X(t)$ to an SDE of the form

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t), \tag{1}$$

where the drift $\mu : \mathbb{R} \to \mathbb{R}$ and diffusion coefficient $\sigma : \mathbb{R} \to \mathbb{R}$ are "suffieciently nice" functions. Here, as usual, $B = \{B(t)\}_{t\geq 0}$ denotes a standard BM. By definition, the solution $X = \{X(t)\}_{t\geq 0}$ to (1) satisfies

$$X(t) = X(0) + \int_0^t \mu(X(r))\,dr + \int_0^t \sigma(X(r))\,dB(r) \quad \text{for } t \geq 0. \tag{2}$$

To get a complete solution to (1) we have to specify a random or non-random initial value $X(0)$, as is also the case when dealing with ordinary differential equations (ODE). Random initial values are required to be independent of B.

The solution $X(t)$ will be adapted to the filtration $\mathcal{F}_t = \sigma\{X(0), \sigma(B(r) : r \leq t)\}$, as we only use $X(0)$ and the process values $\{B(r)\}_{0\leq r\leq t}$ of BM together with the non-random coefficient functions μ and σ to build the value of $X(t)$, see (2).

2.1 Markov property

The solution X to the SDE (1) is a *Markov process*, which is to say that

$$\mathbf{P}\{X(t)\in\cdot\,|\mathcal{F}_s^X\}=\mathbf{P}\{X(t)\in\cdot\,|X(s)\}\quad\text{for } s\leq t.$$

Here $\mathcal{F}_s^X=\sigma\{X(r):r\leq s\}$, $s\geq 0$, is the filtration generated by the process X itself.

While a detailed rigorous proof of the Markov property is very complicated, it is rather less complicated to understand from a more heursitic point of view: Using the representation (2) for both $X(t)$ and $X(s)$ we get

$$\begin{aligned}X(t)&=X(0)+\int_0^t\mu(X(r))\,dr+\int_0^t\sigma(X(r))\,dB(r)-X(s)+X(s)\\&=\int_s^t\mu(X(r))\,dr+\int_s^t\sigma(X(r))\,dB(r)+X(s)\\&=\lim\sum_{i=1}^n\mu(X(t_{i-1}))\,(t_i-t_{i-1})+\lim\sum_{i=1}^n\sigma(X(t_{i-1}))\,(B(t_i)-B(t_{i-1}))+X(s),\end{aligned}$$

where $s=t_0<t_1<\ldots<t_n=t$ is a partion of the interval $[s,t]$ that becomes finer and finer in the limit. From this we see the that the only thing from the past $\mathcal{F}_s^X$ that affects the future value $X(t)$ is the value $X(s)=X(t_0)$ of X at time s.

2.2 Markov transition densities

The *transition density function* of a time homogeneous Markov process X is given by

$$p(t,x,y)=\frac{d}{dy}\mathbf{P}\{X(t+s)\leq y\,|\,X(s)=x\}\quad\text{for } s,t>0.$$

In the particular case when X is the solution to the SDE (1), then the transition density satisfies the Kolmogorov backward partial differential equation (PDE)

$$\frac{\partial p(t,x,y)}{\partial t}=\frac{\sigma(x)^2}{2}\frac{\partial^2 p(t,x,y)}{\partial x^2}+\mu(x)\frac{\partial p(t,x,y)}{\partial x},\quad p(t,x,y)\to\delta(x-y)\quad\text{as } t\downarrow 0.\quad(3)$$

One way to try to find the transition density is thus to try to solve this PDE.

Example 2.1. By Section 3.4 in Klebaner's book BM has transition density

$$p_B(t,x,y)=\frac{1}{\sqrt{2\pi\,t}}\exp\left\{-\frac{(x-y)^2}{2\,t}\right\}.$$

Example 2.2. The *Black-Scholes asset price model* from Examples 5.1 and 5.5 in Klebaner's book is the solution X to the SDE

$$dX(t)=r\,X(t)\,dt+\sigma\,X(t)\,dB(t),$$

where the interest rate $r \in \mathbb{R}$ and the volatility $\sigma > 0$ are parameters. The solution is given by

$$X(t) = X(0)\,\exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma\,B(t)\right\} \quad \text{for } t > 0.$$

Note that $X(t)$ is a random perturbation of the solution $x(t) = x(0)\,\mathrm{e}^{rt}$ to the ODE $dx(t) = r\,x(t)\,dt$. The size of σ determines whether X looks just like such a peturbation (for σ small) or if X will deviate significanly from the ODE solution (for σ large).

If we take the logatirhm of the Black-Scholes model $Y(t) = \log(X(t))$, we get

$$Y(t) = Y(0) + \left(r - \frac{\sigma^2}{2}\right)t + \sigma\,B(t) \quad \text{for } t > 0.$$

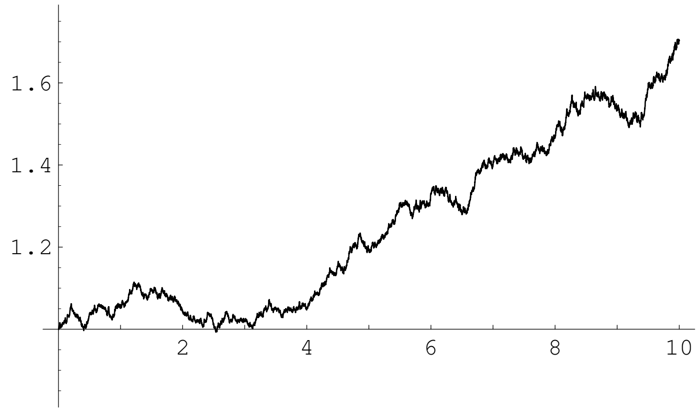
As $Y(0) = \log(X(0))$ is independent of B it follows that

$$\begin{aligned}
&p_Y(t,x,y)\\
&= \frac{d}{dy}\,\mathbf{P}\{Y(t{+}s) \le y \,|\, Y(s) = x\}\\
&= \frac{d}{dy}\,\mathbf{P}\left\{Y(0) + \left(r - \frac{\sigma^2}{2}\right)(t{+}s) + \sigma B(t{+}s) \le y \,\middle|\, Y(0) + \left(r - \frac{\sigma^2}{2}\right)s + \sigma B(s) = x\right\}\\
&= \frac{d}{dy}\,\mathbf{P}\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma\,B(t{+}s) - B(s) + x \le y\right\}\\
&= \frac{d}{dy}\,\mathbf{P}\left\{B(t) \le \frac{1}{\sigma}\left(y - x - \left(r - \frac{\sigma^2}{2}\right)t\right)\right\}\\
&= \frac{1}{\sigma}\,f_{B(t)}\left(\frac{1}{\sigma}\left(y - x - \left(r - \frac{\sigma^2}{2}\right)t\right)\right)\\
&= \frac{1}{\sqrt{2\pi\,t}\,\sigma}\,\exp\left\{-\frac{1}{2\,t\,\sigma^2}\left(y - x - \left(r - \frac{\sigma^2}{2}\right)t\right)^2\right\} \quad \text{for } t > 0.
\end{aligned}$$

We may plot a trajectory $\{X(t)\}_{0 \le t \le 10}$ of the Black-Scholes model with $X(0) = 1$ and $r = \sigma = 0.05$ with 10000 plotgridpoints using Mathematica as

```
In[1]:= x0=1; r=0.05; sigma=0.05; T=10; n=10000; deltat=N[T/n];
        For[i=1; B={0}, i<=n, i++, AppendTo[B,B[[i]]
              + Random[NormalDistribution[0,Sqrt[deltat]]]]];
           X = Table[x0*Exp[(r-sigma^2/2)*i*deltat+sigma*B[[i]]],
              {i,1,1+n}];

In[2]:= Display["~/user/courses/StokAnal/AppliedLecture/BS1.eps",
           ListPlot[X, PlotJoined->True, PlotRange->{0.81,1.79},
```

```
Ticks->{{{1000,""},{2000,"2"},{3000,""},{4000,"4"},
    {5000,""},{6000,"6"},{7000,""},{8000,"8"},
    {9000,""},{10000,"10"}},Automatic}],"EPS"];
```



Example 2.3. An *OU process* $\{X(t)\}_{t\geq 0}$ is the solution to the Langevin SDE

$$dX(t) = -\mu\, X(t)\, dt + \sigma\, dB(t),$$

where $\mu > 0$ (the rate of mean reversion) and $\sigma > 0$ (the volatility) are parameters. This process is basically a scaled (in size by the factor σ) BM, but with a *mean reversion* component $-\mu\, X(t)\, dt$ that takes down the solution X towards zero as soon as X gets too large, and on the other hand takes up X towards zero as soon as X gets too small (/negative). Thus we will have a "balanced development" of the solution so that it never goes away too far from zero.

The transition density for this process is given by

$$p_X(t,x,y) = \frac{\sqrt{\mu}}{\sqrt{\pi\,(1-\mathrm{e}^{-2\,\mu\, t})}\,\sigma}\,\exp\left\{-\frac{\mu\,(y-x\,\mathrm{e}^{-\mu\, t})^2}{\sigma^2\,(1-\mathrm{e}^{-2\,\mu\, t})}\right\} \quad \text{for } t>0.$$

This can be verified by solving the Kolmogorov backward PDE (3) with $\sigma(x) = \sigma$ and $\mu(x) = -\mu\, x$. Indeed, we may use Mathematica to check that p_X satisfies (3):

```
In[3]:= Clear[x,y,t,mu,sigma,pOU]; pOU[mu_,sigma_,x_,y_,t_]
        := (Sqrt[mu]/(Sqrt[2*Pi*(1-Exp[-2*mu*t])]*sigma)) * Exp[
          -(mu*(y-x*Exp[-mu*t])^2)/(sigma^2*(1-Exp[-2*mu*t]))];
In[4]:= Simplify[D[pOU[mu,sigma,x,y,t],t]
          - sigma^2*D[D[pOU[mu,sigma,x,y,t],x],x]/2
          + mu*x*D[pOU[mu,sigma,x,y,t],x]]
```

```
Out[4]= 0
```

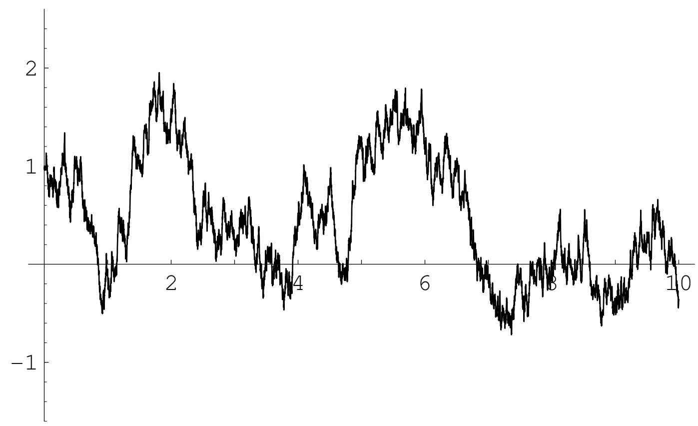
In addition it is easy to see that $p_X(t,x,y) \to 0$ as $t \downarrow 0$ for $x \neq y$ and that p_X really is a density function, as

```
In[5]:= Integrate[pOU[mu,sigma,x,y,t],{y,-Infinity,Infinity},
            Assumptions->sigma>0&&t>0&&mu>0]

Out[5]= 1
```

This establishes the second part of (3) that $p_X(t,x,y) \to \delta(x-y)$ as $t \downarrow 0$.

We may plot an approximative trajectory $\{X(t)\}_{0 \le t \le 10}$ of an OU process with $X(0) = 1$ and $\mu = \sigma = 1$ with 10000 plotgridpoints using Mathematica as

```
In[6]:= x0=1; mu=1; sigma=1; T=10; n=10000; deltat=N[T/n];
        deltaB = Table[Random[NormalDistribution[0, Sqrt[deltat]]],
            {i,1,n}];
        For[i=1; X={x0}, i<=n, i++, AppendTo[X,X[[i]]
                - mu*X[[i]]*deltat + sigma*deltaB[[i]]]];

In[7]:= Display["~/user/courses/StokAnal/AppliedLecture/OU1.eps",
        ListPlot[X, PlotJoined->True, PlotRange->{-1.6, 2.6},
            Ticks->{{{1000,""},{2000,"2"},{3000,""},{4000,"4"},
                {5000,""},{6000,"6"},{7000,""},{8000,"8"},
                {9000,""},{10000,"10"}},Automatic}],"EPS"];
```



2.3 Likelihood functions

By the Markov property the joint density function of the values $X(t_0),\ldots,X(t_n)$ of the solution X to the SDE (1) at times $0\leq t_0<\ldots<t_n$ for $X(t_0),\ldots,X(t_n)$ is given by

$$\begin{aligned}
&f_{X(t_0),\ldots,X(t_n)}(x_0,\ldots,x_n)\\
&= \frac{f_{X(t_0),\ldots,X(t_n)}(x_0,\ldots,x_n)}{f_{X(t_0),\ldots,X(t_{n-1})}(x_0,\ldots,x_{n-1})}\, f_{X(t_0),\ldots,X(t_{n-1})}(x_0,\ldots,x_{n-1})\\
&= f_{X(t_n)\,|\,X(t_0),\ldots,X(t_{n-1})}(x_n\,|\,x_0,\ldots,x_{n-1})\, f_{X(t_0),\ldots,X(t_{n-1})}(x_0,\ldots,x_{n-1})\\
&= f_{X(t_n)\,|\,X(t_{n-1})}(x_n\,|\,x_{n-1})\, f_{X(t_0),\ldots,X(t_{n-1})}(x_0,\ldots,x_{n-1})\\
&= p_X(t_n-t_{n-1},x_{n-1},x_n)\, f_{X(t_0),\ldots,X(t_{n-1})}(x_0,\ldots,x_{n-1})\\
&\qquad\vdots\\
&= p_X(t_n-t_{n-1},x_{n-1},x_n)\, p_X(t_{n-1}-t_{n-2},x_{n-2},x_{n-1})\, f_{X(t_0),\ldots,X(t_{n-2})}(x_0,\ldots,x_{n-2})\\
&\qquad\vdots\\
&= \left(\prod_{i=1}^{n} p_X(t_i-t_{i-1},x_{i-1},x_i)\right) f_{X(t_0)}(x_0).
\end{aligned}$$

If we have a parametric SDE, for examples, of the types considered in Examples 2.2 and 2.3, where the parameters are unknown, then we may estimate the parameter values from real-world observations $x_0,\ldots,x_n$ of the process values $X(t_0),\ldots,X(t_n)$ by means of the so called *maximum likelihood method*. This means that we insert the observed values $x_0,\ldots,x_n$ in the above expression for their joint density, which gives us the so called *likelihood function* $f_{X(t_0),\ldots,X(t_n)}(x_0,\ldots,x_n)$. The value of this function in turn will depend on the unknown parameters. We then estimate the parameters by means of the parameter values that maximizes the likelihood function.

The maximum likelihood method we have desribed above is the usual maximum likelihood method in statitical science, with the only difference that our data are not independent, so that the likelihood function takes on a more complicated form than in the more common setting with independent data. Note that it is often convenient to take the logarithm of the likelihood function

$$\log f_{X(t_0),\ldots,X(t_n)}(x_0,\ldots,x_n) = \sum_{i=1}^{n} \log p_X(t_i-t_{i-1},x_{i-1},x_i) + \log f_{X(t_0)}(x_0)$$

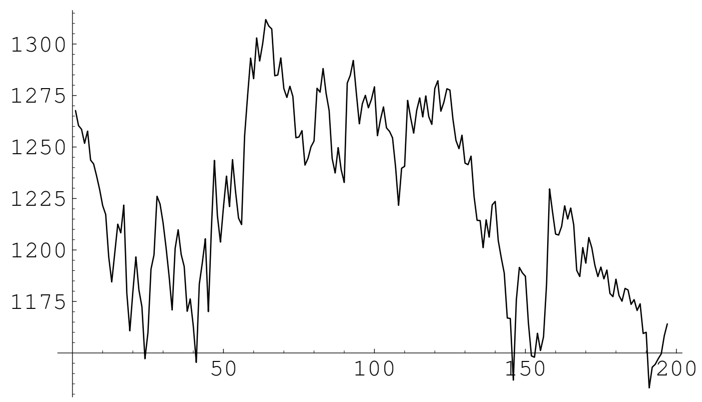
before the maximization is carried out, in order to avoid numerical overflows or underflows in the computer when there are many data observations n.

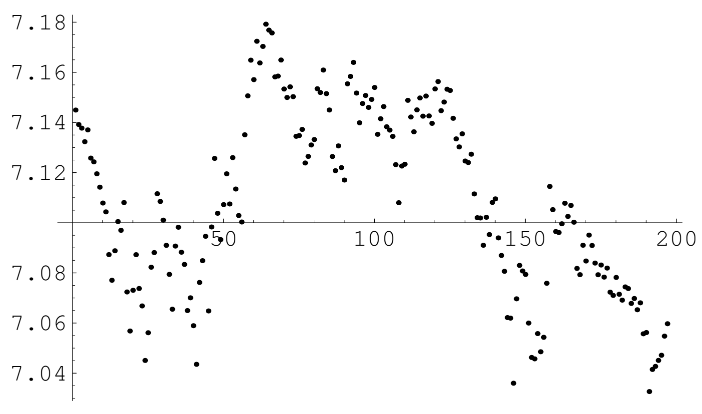
3 Stockholm Stock Exchange index

We use the Black-Scholes model to model the index OMXS30 from January 1, 2007 to October 10, 2007 of the 30 most traded in at the Stockholm Stock Exchange.

Our data set is as follows

```
In[8]:= OMXS30=Reverse[Import[
            "~/user/courses/StokAnal/AppliedLecture/OMXS30","Table"]];
        lt=Length[OMXS30]; OMXS30=Table[OMXS30[[i]][[2]], {i,1,lt}];
        LogOMXS30=Table[Log[OMXS30[[i+1]]]-Log[OMXS30[[i]]], {i,1,lt-1}];
        lt = Length[LogOMXS30];
        Display["~/user/courses/StokAnal/AppliedLecture/OMXS30.eps",
            ListPlot[OMXS30, PlotJoined->True], "EPS"];
        Display["~/user/courses/StokAnal/AppliedLecture/LogOMXS30.eps",
            ListPlot[LogOMXS30];, "EPS"];
```



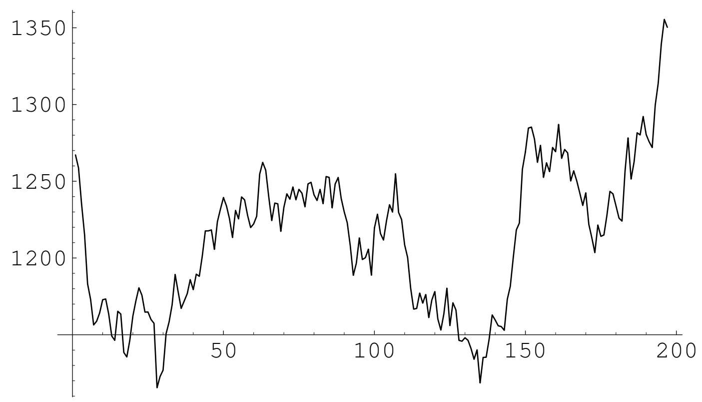


We carry out the maximum likelihood fit by Mathematica (cf. Example 2.2):

```
In[9]:= pY[r_,sigma_,t_,x_,y_] := (1/(Sqrt[2*Pi*t]*sigma))*
            Exp[-(1/(2*t*sigma^2))*(y-x-(r-sigma^2/2)*t)^2];
        Clear[r,sigma];
        NMaximize[{Sum[Log[pY[r,sigma,1,LogOMXS30[[i]],
            LogOMXS30[[i+1]]]], {i,1,lt-1}], sigma>0}, {r,sigma}]
Out[9]= {581.488, {r->-0.000357341, sigma->0.012454}}
```

In a thorough statistical investigation we should have checked the quality of the fit by means of statistical methodology. However, as this is a course in stochastic calculus rather than a course in statistics, we omit such a statistical investigationleave for the moment. Instead we just plot the model with the fitted parameters in order to visually check whether it seems to look like the plot of the OMXS30 data.

```
In[10]:= x0=OMSX30[[1]]; r=-0.000357341; sigma=0.012454; n=lt-1; deltat=1;
         For[i=1; B={0}, i<=n, i++, AppendTo[B,B[[i]]
                + Random[NormalDistribution[0,Sqrt[deltat]]]]];
         X = Table[x0*Exp[(r-sigma^2/2)*i*deltat+sigma*B[[i]]],
                {i,1,1+n}];

In[11]:= Display["~/user/courses/StokAnal/AppliedLecture/BS2.eps",
                ListPlot[X, PlotJoined->True],"EPS"];
```

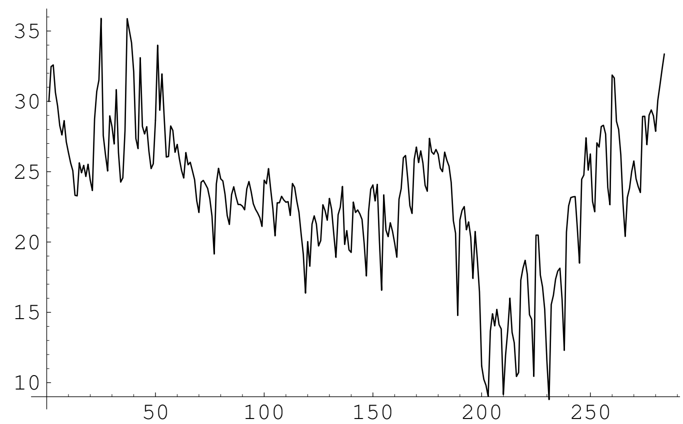


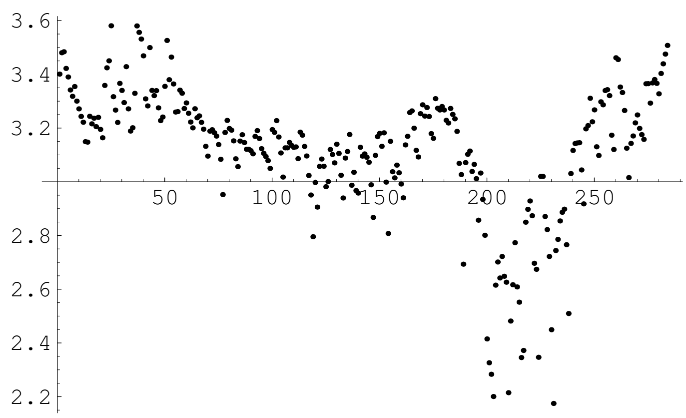
4 Nordpool spot market eletricity prices

We use an exponential (to obtain positive values) OU process to model daily Nordpool spot market eletricity prices from January 1, 2007 to October 11, 2007.

Our data set is as follows

```
In[12]:= Nordpool=Reverse[Import[
             "~/user/courses/StokAnal/AppliedLecture/Nordpool","Table"]];
         lt=Length[Nordpool]; Nordpool=Table[Nordpool[[i]][[2]], {i,1,lt}];
         LogNordpool=Table[Log[Nordpool[[i+1]]]-Log[Nordpool[[i]]],
             {i,1,lt-1}];
         lt = Length[LogNordpool];
         Display["~/user/courses/StokAnal/AppliedLecture/Nordpool.eps",
             ListPlot[Nordpool, PlotJoined->True], "EPS"];
         Display["~/user/courses/StokAnal/AppliedLecture/LogNordpool.eps",
             ListPlot[LogNordpool];, "EPS"];
```





We carry out the maximum likelihood fit by Mathematica (cf. Example 2.3):

```
In[13]:= Clear[mu,sigma];
         NMaximize[{Sum[Log[pOU[mu,sigma,LogNordpool[[i]],
             LogNordpool[[i+1]],1]],{i,1,lt-1}], mu>0, sigma>0},
             {mu,sigma}]
   Out[13]= {17.5628, {mu->0.00468615, sigma->0.0418831}}
```

Again we check the fit just by plotting the model with the fitted parameters. We omit the details of doing such a plot for the moment (see Example 2.3 and Section 3 for hints on how to do it), as such simulations is the topic of next weeks activities.