Exercise session 1, Stochastic Calculus Part I.

1 Let $f(t) = \sin(t)$. Find the variation of f over the interval $[0, 2\pi]$.

Solution. Since f is continuous with continuous derivative, we get $V_f([0, 2\pi]) = \int_0^{2\pi} |f'(s)|ds = \int_0^{2\pi} |cos(s)|ds = \int_0^{\frac{\pi}{2}} cos(s)ds + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -cos(s)ds + \int_{\frac{3\pi}{2}}^{2\pi} cos(s)ds = 4$.

2 Show that $V_{g+h}(t) \leq V_g(t) + V_h(t)$.

Solution. Using the definition of the variation and the triangle inequality, we get $V_{f+g}(t) = \lim_{\delta_n \to 0} \sum_{i=1}^n |f(t_i^n) + g(t_i^n) - f(t_{i-1}^n) - g(t_{i-1}^n)| \leq \lim_{\delta_n \to 0}(\sum_{i=1}^n |f(t_i^n) - f(t_{i-1}^n)| + \sum_{i=1}^n |g(t_i^n) - g(t_{i-1}^n)|) = \lim_{\delta_n \to 0} \sum_{i=1}^n |f(t_i^n) - f(t_{i-1}^n)| + \lim_{\delta_n \to 0} \sum_{i=1}^n |g(t_i^n) - g(t_{i-1}^n)| = V_f(t) + V_g(t)$ where $0 = t_0^n < t_1^n < ... < t_n^n = t$ and $\delta_n = \max_{1 \leq i \leq n} |t_i^n - t_{i-1}^n|$.

3 If $f(t) = e^{-t}$ and $g(t) = [t]$ (the integer part of t), calculate the Stieltjes integrals $\int_0^\infty g(t)df(t)$ and $\int_0^\infty f(t)dg(t)$.

Solution. Since f' exists we get $\int_0^\infty g(t)d f(t) = \int_0^\infty g(t)f'(t)dt = \int_0^\infty [t](-e^{-y})dt = \sum_{n=0}^\infty \int_n^{n+1} n(-e^{-t})dt = \sum_{n=0}^\infty n(e^{-(n+1)} - e^{-n}) = \frac{-1}{e-1}$. For the other integral, note that $g(t) = [t] = \sum_{k=0}^{[t]} h(k)$ where $h(0) = 0$ and $h(k) = 1$ for $k \geq 1$. Hence, $\int_0^\infty f(t)dg(t) = \sum_{k=0}^\infty f(k)h(k) = \sum_{k=1}^\infty e^{-k} = \frac{1}{e-1}$

4 Prove Grönwall's lemma.

Solution. Let $R(t) = \int_0^t h(s)f(s)ds$ and note that R is continuous with $R(0) = 0$. We get $R'(t) = h(t)f(t) \leq h(t)(g(t) + R(t))$, hence $R'(t) - h(t)R(t) \leq h(t)g(t)$. Changing the letter t to s and multiplying both sides with $e^{-\int_t^s h(u)du}$ gives $\frac{d}{ds}(R(s)e^{-\int_t^s h(u)du}) \leq h(s)g(s)e^{-\int_t^s h(u)du}$. Integrating both sides with respect to s from 0 to t gives $R(t)e^{-\int_t^t h(u)du} - R(0)e^{-\int_t^0 h(u)du} \leq \int_0^t h(s)g(s)e^{\int_s^t h(u)du}ds$. But here, the left hand side equals $R(t)$, completing the proof of Grönwall's lemma.