Exercise session 3, Stochastic Calculus Part I.

1 Let $X > 0$. Show that $\mathrm{E}[X] \leq \sum_{n=o}^{\infty} P(X > n)$.

Solution.

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty x\,dF_X(x) = \sum_{n=0}^{\infty} \int_n^{n+1} x\,dF_X(x) \leq \Big\{x \leq n+1,\, \forall x \in [n, n+1]\Big\} \\
&\leq \sum_{n=0}^{\infty} (n+1) \int_n^{n+1} dF_X(x) = \sum_{n=0}^{\infty} (n+1)\big(\mathbb{P}(X \leq n+1) - \mathbb{P}(X \leq n)\big) \\
&= \sum_{n=0}^{\infty} (n+1)\big(\mathbb{P}(X \leq n+1) - 1 + \mathbb{P}(X > n)\big) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(X > n) + \underbrace{\sum_{n=0}^{\infty} -(n+1)\mathbb{P}(X > n+1) + n\mathbb{P}(X > n)}_{=0,\text{ which should be checked.}} = \sum_{n=0}^{\infty} \mathbb{P}(X > n)
\end{aligned}$$

2 Let B_t be a Brownian motion. Show that $X_t = cB(t/c^2)$ is a Brownian motion.

Solution. We check only that the variance is $t - s$. The other properties are almost trivially fulfilled. For $t \geq s$ we get that

$$\mathbb{V}\mathrm{ar}(Y(t) - Y(s)) = \mathbb{V}\mathrm{ar}(c(B(t/c^2) - B(s/c^2))) = c^2 \mathbb{V}\mathrm{ar}(B(t/c^2) - B(s/c^2)) = c^2(t/c^2 - s/c^2) = t - s.$$

3 Let B_t be a Brownian motion. Show that $e^{-\alpha t} B(e^{2\alpha t})$ is a Gaussian process. Find its mean and covariance functions.

Solution. Since B is a Gaussian process, Y is a Gaussian process. The covariance function is now calculated for $s < t$:

$$\rho_Y(s,t) = \mathbb{C}\mathrm{ov}(Y(s), Y(t)) = \mathbb{C}\mathrm{ov}(e^{-\alpha s} B(e^{2\alpha s}), e^{-\alpha t} B(e^{2\alpha t})) = \mathbb{E}[e^{-\alpha s} B(e^{2\alpha s}) e^{-\alpha t} B(e^{2\alpha t})] =$$

$$e^{-\alpha(s+t)} \mathbb{E}[B(e^{2\alpha s}) B(e^{2\alpha t})] = e^{-\alpha(s+t)} \rho_B(e^{2\alpha s}, e^{2\alpha t}) = e^{-\alpha(s+t)} \min(e^{2\alpha s}, e^{2\alpha t}) = e^{-\alpha(s+t)} e^{2\alpha s} = e^{-\alpha|t-s|}.$$

We get the same result for the case $s > t$.

4 Let B_t be a Brownian motion. Show that the process $e^{-t/2} \cosh(B_t)$ is a martingale w.r.t. the filtration $\mathfrak{F}_t = \sigma(B_s, 0 \leq s \leq t)$.

Solution. Let $s > t$. Since B has increments which are independent of the past we get that

$$\begin{aligned}
\mathbb{E}[e^{-s/2}\cosh B(s)|\mathfrak{F}_t] &= \frac{e^{-s/2}}{2}\mathbb{E}[e^{B(s)} + e^{-B(s)}|\mathfrak{F}_t] \\
&= \frac{e^{-s/2}}{2}\mathbb{E}[e^{B(s)-B(t)+B(t)} + e^{-(B(s)-B(t))-B(t)}|\mathfrak{F}_t] \\
&= \frac{e^{-s/2}}{2}\left(e^{B(t)}\mathbb{E}[e^{B(s)-B(t)}|\mathfrak{F}_t] + e^{-B(t)}\mathbb{E}[e^{-(B(s)-B(t))}|\mathfrak{F}_t]\right) \\
&= \frac{e^{-s/2}}{2}\left(e^{B(t)}\mathbb{E}[e^{B(s)-B(t)}] + e^{-B(t)}\mathbb{E}[e^{-(B(s)-B(t))}]\right) \\
&= \left\{\text{Page 50 in Klebaner: } X \sim N(\mu, \sigma^2) \Rightarrow \mathbb{E}[e^{uX}] = e^{\mu u + \frac{\sigma^2 u^2}{2}}\right\} \\
&= \frac{e^{-s/2}}{2}\left(e^{B(t)}e^{(s-t)/2} + e^{-B(t)}e^{(s-t)/2}]\right) \\
&= e^{-t/2}\cosh B(t)
\end{aligned}$$