Exercise session 4, Stochastic Calculus Part I.

1 Let S_n be a zero mean random walk, $\mathrm{E}[\xi_1] = 0$. Let $\sigma^2 = \mathrm{E}[\xi_1^2]$. Show that $S_n^2 - \sigma^2 n$ is a martingale.

Solution. $S_n^2 - \sigma^2 n$ is integrable since

$$\mathbf{E}\{|S_n^2 - \sigma^2 n|\} \leq \mathbf{E}\left\{\left(\sum_{k=1}^{n} \xi_k\right)^2\right\} + \sigma^2 n = \mathbf{E}\left\{\sum_{k=1}^{n} \xi_k^2\right\} + \sigma^2 n = 2\sigma^2 n < \infty.$$

Also, for $m \leq n$, and by properties of conditional expectation

$$\mathbf{E}\{S_n^2 - \sigma^2 n | \mathcal{F}_m\} = \mathbf{E}\{S_n^2 - S_m^2 + S_m^2 - \sigma^2 n | \mathcal{F}_m\} =$$

$$\mathbf{E}\{S_n^2 - S_m^2\} + S_m^2 - \sigma^2 n = \sigma^2 n - \sigma^2 m + S_m^2 - \sigma^2 n = S_m^2 - \sigma^2 m$$

2 Exercise 3.13 in Klebaner.

Solution. See Klebaner 3.9 and 3.13.

3 Let $f(t) = \arctan(t)$. Find $df(B_t)$.

Solution. Direct application of Itô's formula.

4 Let $\tau_1 < \tau_2$ be stopping times. Is $X_t = 1_{(\tau_1, \tau_2]}(t)$ a simple adapted process?

Solution. X_t is adopted since $\{\tau_i < t\}$ is $\mathfrak{F}_t$-measurable. It is not simple, since we can not write it as a step function X_t for fixed t_k. The simple adapted functions

$$X_n(t) = \sum_{k=0}^{n} 1_{\{\tau_1 < \frac{Tk}{n} \leq \tau_2\}}(t) 1_{(\frac{Tk}{n}, \frac{T(k+1)}{n}]}$$

converges to X_t.