

TMS 165/MSA350 Stochastic Calculus Part I Fall 2010

HandIn Number 3, due 8 October

Throughout this handin $B = \{B(t)\}_{t \geq 0}$ denotes Brownian motion.

Task 1. Show that a sequence $\{X_n\}_{n=1}^\infty$ of random variables such that $\mathbf{E}\{X_n^2\} < \infty$ for all n converges in \mathbb{L}^2 to some random variable X if and only if the limit $\lim_{m,n \rightarrow \infty} \mathbf{E}\{X_m X_n\}$ exists. **(10000 points)**

Task 2. Show the isometry property (4.12) in Klebaner's book for the Itô integral process $\{\int_0^t X dB\}_{t \in [0, T]}$ for $X \in E_T$, e.g., using that the property holds for $X \in S_T$ [cf. (4.5) in Klebaner's book]. **(10000 points)**

Task 3. Show that for an $X \in P_T$ we have in the sense of convergence in probability

$$\int_0^T (X_n(t) - X(t))^2 dt \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for some sequence } \{X_n\}_{n=1}^\infty \subseteq S_T,$$

and that the Itô integral process $\{\int_0^t X dB\}_{t \in [0, T]}$ is well-defined as a limit in the sense of convergence in probability of $\int_0^t X_n dB$ as $n \rightarrow \infty$ for $t \in [0, T]$. **(10000 points)**

Task 4. Show that for a process $X \in P_T$ we have

$$\mathbf{P}\left\{\int_0^T X(t)^2 dt = 0\right\} = 1 \Leftrightarrow \mathbf{P}\left\{\int_0^t X dB = 0\right\} = 1 \quad \text{for } t \in [0, T]. \quad \mathbf{(10000 \text{ points})}$$

Task 5. Find stochastic processes $\{X(t)\}_{t \in [0, 1]}$, $\{Y(t)\}_{t \in [0, 1]}$ and $\{Z(t)\}_{t \in [0, 1]}$ that belong to E_1 , $P_1 \setminus E_1$ and P_1^c , respectively. **(10000 points)**

Task 6. Apply Itô's formula Theorem 4.17 in Klebaner's book to $f(X(t), Y(t))$ where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $f(x, y) = g(x)y$ for a two times continuously differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$ and $X = Y = B$. Compare with what you get from applying the integration by parts formula (4.57) in Klebaner's book with $X = g(B)$ and $Y = B$. Derive from the comparison a new proof (without any explicit calculations other than applications of Itô's formula) of the property established in Example 4.23 in Klebaner's book that $[g(B), B](t) = \int_0^t g'(B(s)) ds$ for $t \geq 0$. **(10000 points)**