TMS 165/MSA350 Stochastic Calculus Part I Fall 2010

HandIn Number 5, due 22 October

Througout this handin $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. Consider the CKLS (Chan-Koralyi-Longstaff-Sanders) SDE

$$dX(t) = (\alpha + \beta\, X(t))\, dt + \sigma\, X(t)^{\gamma}\, dB(t) \quad \text{for } t > 0, \quad X(0) = X_0,$$

with parameters $\alpha, \sigma > 0$, $\beta \geq 0$ and $\gamma > 1$, and where X_0 has the stationary distribution (recall Exercise 4 of Exercise Session 5) so that the solution $\{X(t)\}_{t\geq 0}$ is a stationary process (see Task 2 below). Show that the Itô integral part of the solution satisfies

$$\mathbf{E}\bigg\{\int_0^t X(t)^{\gamma}\, dB(t)\bigg\} = C\, t \quad \text{for } t \geq 0,$$

for some strictly negative constant $C < 0$. **(10000 points)**

Task 2. Let $\{X(t)\}_{t\geq 0}$ be a time homogeneous diffusion process that has a stationary distribution and is started according to that stationary distribution at time $t = 0$. Prove that X is a stationary process, which is to say that

$$\mathbf{P}\big\{X(t_1+h) \leq x_1, \ldots, X(t_n+h) \leq x_n\big\} = \mathbf{P}\big\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\big\}$$

for $0 < t_1 < \ldots < t_n$ and $h > 0$. **(10000 points)**

Task 3. Find three SDE that explode, that display transcience but not explosion, and that display recurrence, respectively, but do not feature to exemplify these properties in Klebaner's book. Also, find three SDE where the issue whether the above three mentioned properties hold depends on the starting value of the SDE. **(10000 points)**

Task 4. Find a PDE that is solved by the fair price $p(x,t) = \mathbf{E}\big\{\max\{X(T) - K, 0\}\big| X(t) = x\}$ (for a constant $K > 0$) of an European call option at time $t \in [0,T)$ for an eletricity price $\{X(t)\}_{t\in[0,T]}$ given by the CKLS SDE in Task 1. **(10000 points)**

Task 5. Can an Ornstein-Uhlenbeck process [a solution to the Langevin equation (5.12) in Klebaner's book] become a Brownian motion by means of a change of measure? In that case, how? Otherwise, why not? **(10000 points)**

Task 6. Assume that we have observed the solution $\{X(t)\}_{t\in[0,T]}$ to the CKLS SDE in Task 1 where the γ coefficient is known. Find the estimators of the coefficients α, β and σ according to the methodolgy of Section 10.6 in Klebaner's book. **(10000 points)**