TMS 165/MSA350 Stochastic Calculus Part I Fall 2010

Exercise Session 2, 10 September

Exercise 1. Prove properties (2.17) and (2.21) for conditional expectations.

Solution. It is an easy exercise to see that any constant random variable (that is, a non-random random variable) is measurable wrt. the trivial σ-field $\{\emptyset,\Omega\}$. In particular, $\mathbf{E}\{X\}$ is $\{\emptyset,\Omega\}$-measurable. Further we have

$$\int_\emptyset \mathbf{E}\{X\}\,d\mathbf{P} = 0 = \int_\emptyset X\,d\mathbf{P} \quad \text{and} \quad \int_\Omega \mathbf{E}\{X\}\,d\mathbf{P} = \mathbf{P}\{\Omega\}\,\mathbf{E}\{X\} = \mathbf{E}\{X\} = \int_\Omega X\,d\mathbf{P}.$$

Hence $\mathbf{E}\{X\}$ fulfill the defining properties on page 44 in the book of being the conditional expectation $\mathbf{E}\{X\,|\,\{\emptyset,\Omega\}\}$. This establishes (2.17).

As for (2.21), as $\mathbf{E}\{X\}$ is $\{\emptyset,\Omega\}$-measurable it is measurable wrt. any other σ-field $\mathcal{G}\subseteq\mathcal{F}$ (as any such $\mathcal{G}$ must contain $\{\emptyset,\Omega\}$). For X independent of $\mathcal{G}$ we further have

$$\int_A X\,d\mathbf{P} = \mathbf{E}\{I_A X\} = \mathbf{E}\{I_A\}\,\mathbf{E}\{X\} = \mathbf{P}\{A\}\,\mathbf{E}\{X\} = \int_A \mathbf{E}\{X\}\,d\mathbf{P} \quad \text{for } A\in\mathcal{G}.$$

Hence $\mathbf{E}\{X\}$ fulfill the defining properties of being the conditional expectation $\mathbf{E}\{X\,|\,\mathcal{G}\}$.

Exercise 2. Consider a finite sample space $\Omega=\{1,\ldots,2n\}$ equipped with the σ-field $\mathcal{F}$ consisting of all subsets of Ω together with the uniform probability measure $\mathbf{P}$ on Ω assigning probability $1/(2n)$ to each outcome $\omega\in\Omega$. Calculate $\mathbf{E}\{X\,|\mathcal{G}\}$ for the random variable $X(\omega)=\omega$ and the σ-field $\mathcal{G}=\{\emptyset,A,A^c,\Omega\}$ where $A=\{1,\ldots,n\}$.

Solution. From intuitive reasoning we come up with the hypothesis that

$$\mathbf{E}\{X\,|\mathcal{G}\} = \begin{cases} (n+1)/2 & \text{for } \omega\in A \\ (3n+1)/2 & \text{for } \omega\in A^c \end{cases}.$$

That this really is correct follows from the fact that this random variable is $\mathcal{G}$-measurable and that by elementary calculations together with the uniformity of $\mathbf{P}$ it satisfies

$$\int_B \mathbf{E}\{X\,|\,\mathcal{G}\}\,d\mathbf{P} = \int_B X\,d\mathbf{P} \quad \text{for } B\in\{\emptyset,A,A^c,\Omega\}.$$

Exercise 3. Show that among all zero-mean stochastic processes $\{X(t)\}_{t\geq 0}$ with finite second moments $\mathbf{E}\{X(t)^2\}<\infty$ for $t\geq 0$, the class of martingales contain all processes with independent increments and are all included among processes with uncorrelated increments.

Solution. For X zero-mean with independent increments we have

$$\mathbf{E}\{X(t)|\mathcal{F}_s^X\} = \mathbf{E}\{X(t)-X(s)|\mathcal{F}_s^X\} + \mathbf{E}\{X(s)|\mathcal{F}_s^X\} = \mathbf{E}\{X(t)-X(s)\} + X(s) = X(s)$$

for $s \leq t$, where we use the independent increments and (2.21) together with the fact that X is adapted to the σ-field $\{\mathcal{F}_t^X\}_{t\geq 0}$. Hence X is a martingale.

On the other hand, for X a zero-mean martingale we have

$$\begin{aligned}\mathbf{E}\{(X(u)-X(t))\,(X(s)-X(r))\} &= \mathbf{E}\big\{\mathbf{E}\{(X(u)-X(t))\,(X(s)-X(r))\,|\,\mathcal{F}_s^X\}\big\}\\ &= \mathbf{E}\big\{(X(s)-X(r))\,\mathbf{E}\{X(u)-X(t)\,|\,\mathcal{F}_s^X\}\big\}\\ &= \mathbf{E}\{(X(s)-X(r))\,(X(s)-X(s))\}\\ &= 0\end{aligned}$$

for $0 \leq r \leq s \leq t \leq u$, where we made use of (2.20) and the fact that X is adapted together with (2.18) and the martingale property.

Exercise 4. Prove property (3.4). (Note that it is assumed that $0 < t_1 < \ldots < t_n$ in this formula.)

Solution. We prove (3.4) by induction. Note that the property (3.4) when $n = 1$ is just (3.3), which in turn is a rather elementary formula we proved during Lecture 4.

Now assume that (3.4) holds for $n = k$. Note that (3.4) for $n = k$ in turn means that $(B^x(t_1), \ldots, B^x(t_k))$ has probability density function

$$f_{(B^x(t_1),\ldots,B^x(t_k))}(y_1, \ldots, y_k) = p_{t_1}(x, y_1) \prod_{i=2}^{k} p_{t_i - t_{i-1}}(y_{i-1}, y_i) \quad \text{for } (y_1, \ldots, y_k) \in \mathbb{R}^k.$$

For the case when $n = k+1$ it therefore follows from conditioning on the value $(y_1, \ldots, y_k)$ of $(B^x(t_1), \ldots, B^x(t_k))$ and using independence of increments that

$$\begin{aligned}&\mathbf{P}\bigg\{\bigcap_{i=1}^{k+1} \{B^x(t_i) \leq x_i\}\bigg\}\\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} \mathbf{P}\{B^x(t_{k+1}) - B^x(t_k) + y_k \leq x_{k+1}\}\, f_{(B^x(t_1),\ldots,B^x(t_k))}(y_1, \ldots, y_k)\, dy_1 \ldots dy_k\\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} \Phi\bigg(\frac{x_{k+1} - y_k}{\sqrt{t_{k+1} - t_k}}\bigg)\, p_{t_1}(x, y_1) \prod_{i=2}^{k} p_{t_i - t_{i-1}}(y_{i-1}, y_i)\, dy_1 \ldots dy_k\\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} \int_{-\infty}^{x_{k+1}} p_{t_1}(x, y_1) \prod_{i=2}^{k+1} p_{t_i - t_{i-1}}(y_{i-1}, y_i)\, dy_1 \ldots dy_{k+1}\end{aligned}$$

[as $B^x(t_{k+1}) - B^x(t_k)$ is $\mathrm{N}(0, t_{k+1} - t_k)$-distributed]. This proves (3.4) by induction.

Exercise 5. Let ξ and η be independent standard normal random variables. Show that the process $\{X(t)\}_{t\in\{0,1\}}$ given by $X(0) = \mathrm{sign}(\eta)\,\xi$ and $X(1) = \mathrm{sign}(\xi)\,\eta$ is not Gaussian despite each of the process values $X(0)$ and $X(1)$ are standard Gaussian.

Solution. It is an elementary exercise to see that $X(0)$ and $X(1)$ are standard Gaussian (normal) distributed. Also note that

$$X(0)\,X(1) = \mathrm{sign}(\eta)\,\xi\ \mathrm{sign}(\xi)\,\eta = |\xi|\,|\eta| \geq 0.$$

However, if $(X(0), X(1))$ were bivariate standard Gaussian (as it must be if X is a Gaussian process), then the above non-negativity is possible if and only if $X(0)$ and $X(1)$ have perfect correlation 1. But this is not true as

$$\mathbf{Corr}\{X(0),X(1)\} = \mathbf{Cov}\{X(0),X(1)\} = \mathbf{E}\{X(0)X(1)\} = \mathbf{E}\{|\xi|\,|\eta|\} = (\mathbf{E}\{|\xi|\})^2 = \frac{2}{\pi} < 1$$

by elementary calculations [where we used that $X(0)$ and $X(1)$ are standard Gaussian].

Exercise 6. Prove that the finite dimensional distributions of a zero-mean Gaussian stochastic process $\{X(t)\}_{t\in T}$ are completely characterized by the covariance function of the process.

Solution. Given $t_1, \ldots, t_n \in T$, the distribution of the random variable $(X(t_1), \ldots, X(t_n))$ is determined by its characteristic function (Fourier transform)

$$\mathbf{E}\big\{\mathrm{e}^{i\sum_{j=1}^{n} a_j X(t_j)}\big\} \quad \text{for } (a_1, \ldots, a_n) \in \mathbb{R}^n.$$

As $\sum_{j=1}^{n} a_j X(t_j)$ is a univariate zero-mean Gaussian random variable, that characteristic function in turn is equal to

$$\exp\bigg[-\frac{1}{2}\,\mathbf{Var}\bigg\{\sum_{j=1}^{n} a_j X(t_j)\bigg\}\bigg] = \exp\bigg[-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\, a_j\, \mathbf{Cov}\{X(t_i), X(t_j)\}\bigg],$$

which in turn obviously is determined by the covariance function of X.