TMS 165/MSA350 Stochastic Calculus Part I Fall 2010

Written Exam Tuesday 19 October 8.30 am - 1.30 pm

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AIDS: None.

GRADES: 12000 points (40%) out of the full score 30000 points to pass the exam.

MOTIVATIONS: All answers/solutions must be motivated.

Througout this exam $B = \{B(t)\}_{t\geq 0}$ is a Brownian motion. And Good Luck to you all!

Task 1. For two functions $f, g : [0, \infty) \to \mathbb{R}$ we know the values of the quadratic variations $[f+g, f+g](t)$ and $[f-g, f-g](t)$ at a time $t > 0$. Based on this information only (but knowing nothing more than that about f and g), which of the quantities $[f, f](t)$, $[g, g](t)$ and $[f, g](t)$ can we calculate the value of (if any)? **(5000 points)**

Task 2. Given a fixed time $T > 0$, show that $\{B(t+T) - B(T)\}_{t\geq 0}$ is a Brownian motion. **(5000 points)**

Task 3. Let a function $\mathbb{R}\times[0, \infty) \ni (x, t) \curvearrowright f(x, t) \in \mathbb{R}$ have continuous partial derivatives $\partial f/\partial x$, $\partial f/\partial t$, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x\partial t$ and $\partial^2 f/\partial t^2$ of the first and second order. Under which additional conditions on f is $\{f(B(t), t)\}_{t\geq 0}$ a martingale? **(5000 points)**

Task 4. Remember the definition of the stochastic logarithm $\{\mathcal{L}(X)(t)\}_{t\geq 0}$ of a strictly positive Itô process $\{X(t)\}_{t\geq 0}$ as the solution to the equation

$$dX(t) = X(t)\, d\mathcal{L}(X)(t) \quad \text{for } t > 0, \quad \mathcal{L}(X)(0) = 0.$$

Find the stochastic logarithm of $\{\mathrm{e}^{B(t)}\}_{t\geq 0}$. **(5000 points)**

Task 5. A solution $\{X(t)\}_{t\geq 0}$ to an SDE $dX(t) = \mu(X(t)\, dt + \sigma(X(t))\, dB(t)$ for $t > 0$, $X(0) = X_0$, is a stationary process if X_0 has probability density function

$$f_{X_0}(x) = \frac{1}{\sigma(x)^2} \exp\bigg\{\int_0^x \frac{2\mu(y)}{\sigma(y)^2}\, dy\bigg\} \bigg/ \bigg(\int_{-\infty}^{\infty} \frac{1}{\sigma(z)^2} \exp\bigg\{\int_0^z \frac{2\mu(y)}{\sigma(y)^2}\, dy\bigg\}\, dz\bigg) \quad \text{for } x \in \mathbb{R}.$$

Find an SDE of the type $dX(t) = \sigma(X(t))\, dB(t)$ for $t > 0$, $X(0) = X_0$, the solution of which is a stationary process. **(5000 points)**

Task 6. Write up explicitely the Euler method to solve numerically the Black-Scholes SDE $dX(t) = r\, X(t)\, dt + \sigma\, X(t)\, dB(t)$ for $t > 0$, $X(0) = X_0$, where $r \in \mathbb{R}$ and $\sigma > 0$ are constants. What convergence properties can be expected? **(5000 points)**

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Solutions to Written Exam Tuesday 19 October

Task 1. To know the values of $[f{+}g, f{+}g](t)$ and $[f{-}g, f{-}g](t)$ is equivalent to know the values of $\frac{1}{4}\big([f{+}g, f{+}g](t)-[f{-}g, f{-}g](t)\big) = [f,g](t)$ and $\frac{1}{2}\big([f{+}g, f{+}g](t)+[f{-}g, f{-}g](t)\big)$ $= [f,f](t)+[g,g](t)$. Hence we can say what is the value of $[f,g](t)$, but not what is the values of $[f,f](t)$ or $[g,g](t)$ (as we know the value of the sum of these two only).

Task 2. Since $\{B(t+T)-B(T)\}_{t\geq 0}$ is a continuous process (as a function of t) with $(B(t{+}T){-}B(T))-(B(s{+}T){-}B(T)) = B(t{+}T){-}B(s{+}T)$ $\mathrm{N}(0,t{-}s)$ distributed for $0\leq s\leq t$, we need only show that the process has independents increments. That this in turn is the case follows from noting that $(B(t{+}T){-}B(T))-(B(s{+}T){-}B(T)) = B(t{+}T){-}B(s{+}T)$ is independent of $\mathcal{F}^B_{s+T} = \sigma(B(u): u\leq s{+}T)$ for $0\leq s\leq t$, and therefore in particular independent of $\sigma(B(u{+}T)-B(T): u\leq s)\subseteq \mathcal{F}^B_{s+T}$. Alternativley, it is enough to check that $\{B(t+T)-B(T)\}_{t\geq 0}$ is a zero-mean Gaussian process with the same covariance function as Brownian motion.

Task 3. Since Itô's formula shows that

$$f(B(t),t) = f(B(0),0) + \int_0^t \bigg(\frac{\partial f}{\partial t}(B(s),s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(B(s),s)\bigg)\,ds + \int_0^t \frac{\partial f}{\partial x}(B(s),s)\,dB(s)$$

for $t\geq 0$, we see that $\{f(B(t),t)\}_{t\geq 0}$ is a martingale if

$$\frac{\partial f(x,t)}{\partial t}+\frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2} = 0 \quad \text{and} \quad \int_0^t \mathbf{E}\bigg\{\Big(\frac{\partial f}{\partial x}(B(s),s)\Big)^2\bigg\}\,ds<\infty \quad \text{for } (x,t)\in\mathbb{R}{\times}[0,\infty).$$

Task 4. As the stochastic exponential of $\{\mathcal{L}(\mathrm{e}^B)(t)\}_{t\geq 0}$ must be $\{\mathrm{e}^{B(t)}\}_{t\geq 0}$ (by an inspection of the definition of stochastic logarithm), we see that $\mathcal{L}(\mathrm{e}^B)(t) = B(t)-t/2$.

Task 5. We may take $\sigma(x) = \max\{1,|x|\}$ and $f_{X_0}(x) = 1/(4\max\{1,|x|^2\})$.

Task 6. See Stig Larsson's lecture notes.