TMS 165/MSA350 Stochastic Calculus Part I Fall 2011

Home exercises for Chapter 3 in Klebaner's book

Througout this set of exercises $B = \{B(t)\}_{t \geq 0}$ denotes Brownian motion.

Task 1. Show that the stochastic process $\{B(t)^4 - 6tB(t)^2 + 3t^2\}_{t \geq 0}$ is a martingal with respect to the filtration $\{\mathcal{F}_t^B\}_{t \geq 0}$ generated by B.

Task 2. For an $\varepsilon > 0$, consider the differential ratio process $\Delta_\varepsilon = \{\Delta_\varepsilon(t)\}_{t \geq 0}$ given by

$$\Delta_\varepsilon(t) = \frac{B(t+\varepsilon) - B(t)}{\varepsilon} \quad \text{for } \ t \geq 0.$$

Show that the covariance function

$$r_\varepsilon(t) = \mathbf{Cov}\{\Delta_\varepsilon(s), \Delta_\varepsilon(s+t)\}$$

of Δ_ε is a triangle like function that depends on the difference t between $s \geq 0$ and $s+t \geq 0$ only. Show that $r_\varepsilon(t) \to \delta(t)$ (Dirac's δ-function) as $\varepsilon \downarrow 0$. Simulate a sample path of $\{\Delta_\varepsilon(t)\}_{t \in [0,1]}$ for a really small $\varepsilon > 0$ and plot it graphically. Discuss the claim that the (non-existing in the usual sense) derivative process $\{B'(t)\}_{t \geq 0}$ of B is white noise.

Task 3. Nobert Wiener (1894-1964) defined the stochastic integral process $\{\int_0^t g\, dB\}_{t \geq 0}$ with respect to B for continuously differentiable functions $g : [0, \infty) \to \mathbb{R}$ as

$$\int_0^t g\, dB = g(t)B(t) - \int_0^t B\, dg = g(t)B(t) - \int_0^t B(r)g'(r)\, dr \quad \text{for } \ t \geq 0.$$

[Of course, the motivation for this definition comes from the integration by parts formula Equation 1.20 in Klebaner's book.] Show by means of direct calculation (not using Itô's formula) that $\{\int_0^t g\, dB\}_{t \geq 0}$ defined in this way is a martingale.

Task 4. As B has strictly positive quadratic variation and is continuous, B must have infinite variation V_B by Theorem 1.10 in Klebaner's book. Another way to understand that $V_B(t) = \infty$ for $t > 0$ is the following: For increasingly fine partitions $0 = t_0 < t_1 < \ldots < t_n = t$ of the interval $[0, t]$, compute the limits of

$$\mathbf{E}\left\{\sum_{i=1}^{n} |B(t_i) - B(t_{i-1})|\right\} \qquad \text{and} \qquad \mathbf{Var}\left\{\sum_{i=1}^{n} |B(t_i) - B(t_{i-1})|\right\}$$

as $\max_{1 \leq i \leq n} t_i - t_{i-1} \downarrow 0$. Explain how to conclude that $V_B(t) = \infty$.