

TMS 165/MSA350 Stochastic Calculus Part I Fall 2011

Home exercises for Chapters 6 and 10 in Klebaner's book

Throughout this set of exercises $B = \{B(t)\}_{t \geq 0}$ denotes Brownian motion.

Task 1. Consider the CKLS (Chan-Koralyi-Longstaff-Sanders) SDE

$$dX(t) = (\alpha + \beta X(t)) dt + \sigma X(t)^\gamma dB(t) \quad \text{for } t > 0, \quad X(0) = X_0,$$

with parameters $\alpha, \sigma > 0$, $\beta \geq 0$ and $\gamma > 1$, and where X_0 has the stationary distribution (recall Exercise 4 of Exercise session 5) so that the solution $\{X(t)\}_{t \geq 0}$ is a stationary process (see Task 2 below). Show that the Itô integral part of the solution satisfies

$$\mathbf{E} \left\{ \int_0^t X(t)^\gamma dB(t) \right\} = C t \quad \text{for } t \geq 0,$$

for some strictly negative constant $C < 0$.

Task 2. Let $\{X(t)\}_{t \geq 0}$ be a time homogeneous diffusion process that has a stationary distribution and is started according to that stationary distribution at time $t = 0$. Prove that X is a stationary process, which is to say that

$$\mathbf{P}\{X(t_1+h) \leq x_1, \dots, X(t_n+h) \leq x_n\} = \mathbf{P}\{X(t_1) \leq x_1, \dots, X(t_n) \leq x_n\}$$

for $0 < t_1 < \dots < t_n$ and $h > 0$.

Task 3. Find three SDE that explode, that display transience but not explosion, and that display recurrence, respectively, but do not feature to exemplify these properties in Klebaner's book. Also, find three SDE where the issue whether the above three mentioned properties hold depends on the starting value of the SDE.

Task 4. Find a PDE that is solved by the fair price $p(x, t) = \mathbf{E}\{\max\{X(T) - K, 0\} | X(t) = x\}$ (for a constant $K > 0$) of an European call option at time $t \in [0, T]$ for an electricity price $\{X(t)\}_{t \in [0, T]}$ given by the CKLS SDE in Task 1.

Task 5. Can an Ornstein-Uhlenbeck process [a solution to the Langevin equation 5.12 in Klebaner's book] become a Brownian motion by means of a change of measure? In that case, how? Otherwise, why not?

Task 6. Assume that we have observed the solution $\{X(t)\}_{t \in [0, T]}$ to the CKLS SDE in Task 1 where the γ coefficient is known. Find the estimators of the coefficients α, β and σ according to the methodology of Section 10.6 in Klebaner's book.