TMS 165/MSA350 Stochastic Calculus Part I Fall 2011

Exercise session 1

Exercise 1. Explain why the definition of variation of a function according to Equation 1.7 in Klebaner's book is more generally valid than that according to Equation 1.9.

Solution. Equation 1.7 defines the variation as the supremum of a set of real numbers [namely the set of all values the sum in (1.7) can take for different partitions], and a supremum of a set of real numbers is always a well-defined quantity. Now we may consider the function $g : [0,1] \to \mathbb{R}$ given by $g(\frac{1}{2}) = 1$ and $g(x) = 0$ for $x \in [0,1] \setminus \{\frac{1}{2}\}$, which has a well-defined variation $V_g([0,1]) = 2$ according to (1.7), but which does not have a well-defined variation $V_g([0,1])$ according to (1.9).

Exercise 2. Prove Equations 1.16 and 1.17 in Klebaner's book for quadratic variation.

Solution. Clearly, (1.17) follows readily from using the definition (1.15) of quadratic variation together with a few simple algebraic manipulations. Further, we may derive (1.16) from (1.17) together with symmetry as

$$\begin{aligned}
\frac{1}{2}\big(\lfloor f+g, f+g\rfloor - \lfloor f,f\rfloor - \lfloor g,g\rfloor\big) &= \frac{1}{2}\big(\lfloor f, f+g\rfloor + \lfloor g, f+g\rfloor - \lfloor f,f\rfloor - \lfloor g,g\rfloor\big) \\
&= \frac{1}{2}\big(\lfloor f+g, f\rfloor + \lfloor f+g, g\rfloor - \lfloor f,f\rfloor - \lfloor g,g\rfloor\big) \\
&= \frac{1}{2}\big(\lfloor f,f\rfloor + \lfloor g,f\rfloor + \lfloor f,g\rfloor + \lfloor g,g\rfloor - \lfloor f,f\rfloor - \lfloor g,g\rfloor\big) \\
&= \frac{1}{2}\big(\lfloor g,f\rfloor + \lfloor f,g\rfloor\big) \\
&= \lfloor f,g\rfloor.
\end{aligned}$$

Exercise 3. Calculate the Stieltjes integral of a continuous function $f : [0,\infty) \to \mathbb{R}$ with respect to the function $[0,\infty) \ni x \curvearrowright g(x) = \lfloor x\rfloor \in [0,\infty)$ (i.e., the integer part of x) over an interval $(a,b] \subseteq [0,\infty)$.

Solution. Selecting integers $k, \ell \in \mathbb{N}$ such that $\lfloor a\rfloor = k$ and $\lfloor b\rfloor = \ell$, we either have $k = \ell$, in which case $k \le a < b < k+1$ and $\int_{(a,b]} f\,dg = 0$, or else $k < \ell$, in which case $k \le a < k+1 \le \ell \le b < \ell+1$ and $\int_{(a,b]} f\,dg = \sum_{n=\lfloor a\rfloor+1}^{\lfloor b\rfloor} f(n)$.

Exercise 4. Construct a standard normal distributed random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Solution. With $\Omega = \mathbb{R}$, $\mathcal{F}$ the Borel σ-field generated by the intervals in $\mathbb{R}$ and $\mathbf{P}\{A\} = \int_A \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\, dx$ for $A \in \mathcal{F}$, the random variable $X(\omega) = \omega$ for $\omega \in \Omega$ is standard normal distributed.

Exercise 5. Calculate the expectation $\mathbf{E}\{X^+\}$ using the definition of the Lebesgue integral for the random variable in Exercise 4.

Solution. Using Example 2.9 together with page 33 in Klebaner's book we see that

$$\begin{aligned}
\mathbf{E}\{X^+\} &= \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \frac{k}{2^n}\, \mathbf{P}\big\{\big[\tfrac{k}{2^n}, \tfrac{k+1}{2^n}\big)\big\} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \int_{k/2^n}^{(k+1)/2^n} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2}\, dx \\
&= \lim_{n\to\infty} \sum_{k=1}^{n2^n-1} \frac{k}{2^n}\, \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-(k/2^n)^2/2}\, \frac{1}{2^n} \\
&= \int_0^\infty x\, \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2}\, dx \\
&= 1/\sqrt{2\pi}.
\end{aligned}$$