TMS 165/MSA350 Stochastic Calculus Part I Fall 2011

Exercise session 2

Exercise 1. Prove Equations 2.17 and 2.21 in Klebaner's book for conditional expectations.

Solution. It is an easy exercise to see that any constant random variable (that is, a non-random random variable) is measurable wrt. the trivial σ-field $\{\emptyset, \Omega\}$. In particular, $\mathbf{E}\{X\}$ is $\{\emptyset, \Omega\}$-measurable. Further we have

$$\int_\emptyset \mathbf{E}\{X\}\, d\mathbf{P} = 0 = \int_\emptyset X\, d\mathbf{P} \quad \text{and} \quad \int_\Omega \mathbf{E}\{X\}\, d\mathbf{P} = \mathbf{P}\{\Omega\}\, \mathbf{E}\{X\} = \mathbf{E}\{X\} = \int_\Omega X\, d\mathbf{P}.$$

Hence $\mathbf{E}\{X\}$ fulfill the defining properties on page 44 in the book of being the conditional expectation $\mathbf{E}\{X\,|\,\{\emptyset, \Omega\}\}$. This establishes (2.17).

As for (2.21), as $\mathbf{E}\{X\}$ is $\{\emptyset, \Omega\}$-measurable it is measurable wrt. any other σ-field $\mathcal{G} \subseteq \mathcal{F}$ (as any such $\mathcal{G}$ must contain $\{\emptyset, \Omega\}$). For X independent of $\mathcal{G}$ we further have

$$\int_A X\, d\mathbf{P} = \mathbf{E}\{I_A X\} = \mathbf{E}\{I_A\}\, \mathbf{E}\{X\} = \mathbf{P}\{A\}\, \mathbf{E}\{X\} = \int_A \mathbf{E}\{X\}\, d\mathbf{P} \quad \text{for } A \in \mathcal{G}.$$

Hence $\mathbf{E}\{X\}$ fulfill the defining properties of being the conditional expectation $\mathbf{E}\{X\,|\,\mathcal{G}\}$.

Exercise 2. Consider a finite sample space $\Omega = \{1, \ldots, 2n\}$ equipped with the σ-field $\mathcal{F}$ consisting of all subsets of Ω together with the uniform probability measure $\mathbf{P}$ on Ω assigning probability $1/(2n)$ to each outcome $\omega \in \Omega$. Calculate $\mathbf{E}\{X\,|\mathcal{G}\}$ for the random variable $X(\omega) = \omega$ and the σ-field $\mathcal{G} = \{\emptyset, A, A^c, \Omega\}$ where $A = \{1, \ldots, n\}$.

Solution. From intuitive reasoning we come up with the hypothesis that

$$\mathbf{E}\{X\,|\mathcal{G}\} = \begin{cases} (n+1)/2 & \text{for } \omega \in A \\ (3n+1)/2 & \text{for } \omega \in A^c \end{cases}.$$

That this really is correct follows from the fact that this random variable is $\mathcal{G}$-measurable and that by elementary calculations together with the uniformity of $\mathbf{P}$ it satisfies

$$\int_B \mathbf{E}\{X\,|\,\mathcal{G}\}\, d\mathbf{P} = \int_B X\, d\mathbf{P} \quad \text{for } B \in \{\emptyset, A, A^c, \Omega\}.$$

Exercise 3. Show that among all zero-mean stochastic processes $\{X(t)\}_{t\geq 0}$ with finite second moments $\mathbf{E}\{X(t)^2\} < \infty$ for $t \geq 0$, the class of martingales contain all processes with independent increments and are all included among processes with uncorrelated increments.

Solution. For X zero-mean with independent increments we have

$$\mathbf{E}\{X(t)|\mathcal{F}_s^X\} = \mathbf{E}\{X(t)-X(s)|\mathcal{F}_s^X\} + \mathbf{E}\{X(s)|\mathcal{F}_s^X\} = \mathbf{E}\{X(t)-X(s)\} + X(s) = X(s)$$

for $s \leq t$, where we use the independent increments and (2.21) together with the fact that X is adapted to the σ-field $\{\mathcal{F}_t^X\}_{t\geq 0}$. Hence X is a martingale.

On the other hand, for X a zero-mean martingale we have

$$\begin{aligned}\mathbf{E}\{(X(u)-X(t))\,(X(s)-X(r))\} &= \mathbf{E}\big\{\mathbf{E}\{(X(u)-X(t))\,(X(s)-X(r))\,|\,\mathcal{F}_s^X\}\big\}\\ &= \mathbf{E}\big\{(X(s)-X(r))\,\mathbf{E}\{X(u)-X(t)\,|\,\mathcal{F}_s^X\}\big\}\\ &= \mathbf{E}\{(X(s)-X(r))\,(X(s)-X(s))\}\\ &= 0\end{aligned}$$

for $0 \leq r \leq s \leq t \leq u$, where we made use of Equation 2.20 in Klebaner's book and the fact that X is adapted together with Equation 2.18 and the martingale property.

Exercise 4. Prove Equation 3.4 in Klebaner's book. (Note that it is assumed that $0 < t_1 < \ldots < t_n$ in this formula.)

Solution. We prove (3.4) by induction. Note that the property (3.4) when $n = 1$ is just (3.3), which in turn is a rather elementary formula we proved during Lecture 4.

Now assume that (3.4) holds for $n = k$. Note that (3.4) for $n = k$ in turn means that $(B^x(t_1), \ldots, B^x(t_k))$ has probability density function

$$f_{(B^x(t_1),\ldots,B^x(t_k))}(y_1, \ldots, y_k) = p_{t_1}(x, y_1) \prod_{i=2}^{k} p_{t_i - t_{i-1}}(y_{i-1}, y_i) \quad \text{for } (y_1, \ldots, y_k) \in \mathbb{R}^k.$$

For the case when $n = k+1$ it therefore follows from conditioning on the value $(y_1, \ldots, y_k)$ of $(B^x(t_1), \ldots, B^x(t_k))$ and using independence of increments that

$$\begin{aligned}&\mathbf{P}\Big\{\bigcap_{i=1}^{k+1}\{B^x(t_i) \leq x_i\}\Big\}\\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} \mathbf{P}\{B^x(t_{k+1}) - B^x(t_k) + y_k \leq x_{k+1}\}\, f_{(B^x(t_1),\ldots,B^x(t_k))}(y_1, \ldots, y_k)\, dy_1 \ldots dy_k\\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} \Phi\bigg(\frac{x_{k+1} - y_k}{\sqrt{t_{k+1} - t_k}}\bigg)\, p_{t_1}(x, y_1) \prod_{i=2}^{k} p_{t_i - t_{i-1}}(y_{i-1}, y_i)\, dy_1 \ldots dy_k\\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} \int_{-\infty}^{x_{k+1}} p_{t_1}(x, y_1) \prod_{i=2}^{k+1} p_{t_i - t_{i-1}}(y_{i-1}, y_i)\, dy_1 \ldots dy_{k+1}\end{aligned}$$

[as $B^x(t_{k+1})-B^x(t_k)$ is $\mathrm{N}(0,t_{k+1}-t_k)$-distributed]. This proves (3.4) by induction.

Exercise 5. Let ξ and η be independent standard normal random variables. Show that the process $\{X(t)\}_{t\in\{0,1\}}$ given by $X(0)=\mathrm{sign}(\eta)\,\xi$ and $X(1)=\mathrm{sign}(\xi)\,\eta$ is not Gaussian despite each of the process values $X(0)$ and $X(1)$ are standard Gaussian.

Solution. It is an elementary exercise to see that $X(0)$ and $X(1)$ are standard Gaussian (normal) distributed. Also note that

$$X(0)\,X(1)=\mathrm{sign}(\eta)\,\xi\;\mathrm{sign}(\xi)\,\eta=|\xi|\,|\eta|\geq 0.$$

However, if $(X(0),X(1))$ were bivariate standard Gaussian (as it must be if X is a Gaussian process), then the above non-negativity is possible if and only if $X(0)$ and $X(1)$ have perfect correlation 1. But this is not true as

$$\mathbf{Corr}\{X(0),X(1)\}=\mathbf{Cov}\{X(0),X(1)\}=\mathbf{E}\{X(0)X(1)\}=\mathbf{E}\{|\xi|\,|\eta|\}=(\mathbf{E}\{|\xi|\})^2=\frac{2}{\pi}<1$$

by elementary calculations [where we used that $X(0)$ and $X(1)$ are standard Gaussian].

Exercise 6. Prove that the finite dimensional distributions of a zero-mean Gaussian stochastic process $\{X(t)\}_{t\in T}$ are completely characterized by the covariance function of the process.

Solution. Given $t_1,\ldots,t_n\in T$, the distribution of the random variable $(X(t_1),\ldots,X(t_n))$ is determined by its characteristic function (Fourier transform)

$$\mathbf{E}\big\{\mathrm{e}^{i\sum_{j=1}^n a_jX(t_j)}\big\}\quad\text{for}\;\;(a_1,\ldots,a_n)\in\mathbb{R}^n.$$

As $\sum_{j=1}^n a_jX(t_j)$ is a univariate zero-mean Gaussian random variable, that characteristic function in turn is equal to

$$\exp\bigg[-\frac{1}{2}\,\mathbf{Var}\bigg\{\sum_{j=1}^n a_jX(t_j)\bigg\}\bigg]=\exp\bigg[-\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_i\,a_j\,\mathbf{Cov}\{X(t_i),X(t_j)\}\bigg],$$

which in turn obviously is determined by the covariance function of X.