

TMS 165/MSA350 Stochastic Calculus Part I Fall 2011

Exercise session 5

Throughout this exercise session $B = \{B(t)\}_{t \geq 0}$ denotes Brownian motion.

Exercise 1. Show that

$$X(t) = e^{-\alpha t} \left(\frac{\sigma}{\sqrt{2\alpha}} (B(e^{2\alpha t}) - B(1)) + x_0 \right) \quad \text{for } t \geq 0$$

is an Ornstein-Uhlenbeck process in the sense that it got the same distributional properties (finite dimensional distributions) as the solution

$$\{X(t)\}_{t \geq 0} = \left\{ e^{-\alpha t} \left(x_0 + \sigma \int_0^t e^{\alpha r} dB(r) \right) \right\}_{t \geq 0}$$

to the Langevin SDE

$$dX(t) = -\alpha X(t) dt + \sigma dB(t) \quad \text{for } t > 0, \quad X(0) = x_0,$$

where $\alpha, \sigma > 0$ and $x_0 \in \mathbb{R}$ are constants.

Solution. As both the above X processes are Gaussian they have the same finite dimensional distributions if their mean and covariance functions agree. Here we clearly have $\mathbf{E}\{X(t)\} = e^{-\alpha t} x_0$ for $t \geq 0$ for both the X processes. Further, we have

$$\begin{aligned} \mathbf{Cov}\{X(s), X(t)\} &= \frac{\sigma^2}{2\alpha} e^{-\alpha(s+t)} \mathbf{Cov}\{B(e^{2\alpha s}) - B(1), B(e^{2\alpha t}) - B(1)\} \\ &= \frac{\sigma^2}{2\alpha} e^{-\alpha(s+t)} (e^{2\alpha \min\{s,t\}} - 1 - 1 + 1) \\ &= \frac{\sigma^2}{2\alpha} (e^{-\alpha|s-t|} - e^{-\alpha(s+t)}) \quad \text{for } s, t \geq 0 \end{aligned}$$

for the first X process, while Theorem 4.11 in Klebaner's book shows that

$$\mathbf{Cov}\{X(s), X(t)\} = \sigma^2 e^{-\alpha(s+t)} \int_0^{\min\{s,t\}} e^{2\alpha r} dr = \frac{\sigma^2}{2\alpha} (e^{-\alpha|s-t|} - e^{-\alpha(s+t)})$$

for $s, t \geq 0$ for the second X process.

Exercise 2. Use the expression for an Ornstein-Uhlenbeck process expressed in terms of B from Exercise 1 to find the transition density function for the solution to the Langevin SDE (the Ornstein-Uhlenbeck process).

Solution. We have

$$\begin{aligned} X(t+s) &= e^{-\alpha(t+s)} \left(\frac{\sigma}{\sqrt{2\alpha}} (B(e^{2\alpha(t+s)}) - B(1)) + x_0 \right) \\ &= e^{-\alpha(t+s)} x_0 + \frac{\sigma}{\sqrt{2\alpha}} e^{-\alpha(t+s)} \left((B(e^{2\alpha(t+s)}) - B(e^{2\alpha s})) + (B(e^{2\alpha s}) - B(1)) \right) \\ &= \frac{\sigma}{\sqrt{2\alpha}} e^{-\alpha(t+s)} (B(e^{2\alpha(t+s)}) - B(e^{2\alpha s})) + e^{-\alpha t} X(s), \end{aligned}$$

where

$$\frac{\sigma}{\sqrt{2\alpha}} e^{-\alpha(t+s)} (B(e^{2\alpha(t+s)}) - B(e^{2\alpha s}))$$

is an $N(0, (\sigma^2/(2\alpha))(1-e^{-2\alpha t}))$ -distributed random variable independent of $\{X(r)\}_{r \leq s}$.

It follows that $(X(t+s)|X(s)=x)$ is $N(e^{-\alpha t}x, (\sigma^2/(2\alpha))(1-e^{-2\alpha t}))$ -distributed, so that

$$p(y, t+s, x, s) = \frac{d}{dy} P(y, t+s, x, s) = \frac{\sqrt{\alpha}}{\sqrt{\pi(1-e^{-2\alpha t})}\sigma} \exp\left\{-\frac{\alpha(y-xe^{-\alpha t})^2}{\sigma^2(1-e^{-2\alpha t})}\right\}$$

for $t+s > s \geq 0$ and $x, y \in \mathbb{R}$.

Exercise 3. Solve the Stratanovich SDE

$$dX(t) = -\alpha dt + \sigma X(t) \partial B(t) \quad \text{for } t > 0, \quad X(0) = x_0,$$

where $\alpha, \sigma > 0$ and $x_0 \in \mathbb{R}$ are constants.

Solution. By Theorem 5.20 in Klebaner's book the above SDE is equivalent to the Itô SDE

$$dX(t) = \left(\frac{1}{2}\sigma^2 X(t) - \alpha\right) dt + \sigma X(t) dB(t) \quad \text{for } t > 0, \quad X(0) = x_0.$$

This in turn is a rather simple form of the linear SDE treated in Section 5.3 in Klebaner's book, with a solution given by Equations 5.25 together with 5.30 in Klebaner's book as

$$X(t) = U(t) \left(x_0 - \alpha \int_0^t \frac{ds}{U(s)} \right) \quad \text{where } U(t) = e^{\sigma B(t)},$$

which is to say that

$$X(t) = x_0 e^{\sigma B(t)} - \alpha e^{\sigma B(t)} \int_0^t e^{-\sigma B(s)} ds \quad \text{for } t \geq 0.$$

Exercise 4. The CKLS (Chan-Koralyi-Longstaff-Sanders) SDE is given by

$$dX(t) = (\alpha + \beta X(t)) dt + \sigma X(t)^\gamma dB(t) \quad \text{for } t > 0, \quad X(0) = x_0,$$

where $\alpha, \sigma, \gamma, x_0 > 0$ and $\beta \in \mathbb{R}$ are constants. This SDE is used in contemporary mathematical finance research as a model for, e.g., interest rates and/or deseasonalized electricity prices, and is famous for being very hard to do inference for and very hard to simulate when $\gamma > 1$. Determine the stationary distribution for this SDE when it exists.

Solution. First note that the fact that $\alpha, x_0 > 0$ ensures that the solution is strictly positive when it exists. From Equation 6.69 in Klebaner's book we further see that the stationary probability density function is given by

$$\pi(x) = \frac{1}{C x^{2\gamma}} \exp\left\{\int_1^x \frac{2(\alpha + \beta y)}{\sigma^2 y^{2\gamma}} dy\right\} \quad \text{for } x > 0,$$

whenever this function can be normalized to become a density, that is, whenever

$$C = \int_0^\infty \frac{1}{x^{2\gamma}} \exp\left\{\int_1^x \frac{2(\alpha + \beta y)}{\sigma^2 y^{2\gamma}} dy\right\} dx < \infty.$$

The issue whether C is finite or not in turn clearly boils down to check the integrability properties of the function

$$f(x) = \frac{1}{x^{2\gamma}} \exp\left\{\int_1^x \frac{2(\alpha + \beta y)}{\sigma^2 y^{2\gamma}} dy\right\}$$

as $x \downarrow 0$ and as $x \uparrow \infty$. Now, as $x \downarrow 0$ we see that

$$f(x) \sim \begin{cases} C_1 x^{-2\gamma} & \text{for } \gamma \in (0, 1/2), \\ C_2 x^{2\alpha/\sigma^2 - 1} & \text{for } \gamma = 1/2, \\ C_3 x^{-2\gamma} \exp\{-(2\alpha/(\sigma^2(2\gamma-1)))x^{-(2\gamma-1)}\} & \text{for } \gamma > 1/2, \end{cases}$$

where $C_1, C_2, C_3 > 0$ are constants. This is to say that we always have the integrability required as $x \downarrow 0$. When $x \uparrow \infty$ we further see that

$$f(x) \sim \begin{cases} C_4 x^{-2\gamma} & \text{for } \gamma > 1, \\ C_5 x^{-2+2\beta/\sigma^2} & \text{for } \gamma = 1, \\ C_6 x^{-2\gamma} \exp\{(\beta/(\sigma^2(1-\gamma)))x^{2-2\gamma}\} & \text{for } \gamma \in (1/2, 1), \\ C_7 x^{2\alpha/\sigma^2 - 1} \exp\{(2\beta/\sigma^2)x\} & \text{for } \gamma = 1/2, \\ C_8 x^{-2\gamma} \exp\{(\beta/(\sigma^2(1-\gamma)))x^{2-2\gamma} + (2\alpha/(\sigma^2(1-2\gamma)))x^{1-2\gamma}\} & \text{for } \gamma \in (0, 1/2), \end{cases}$$

where $C_4, \dots, C_8 > 0$ are constants. This is to say that we have the integrability required when

$$\gamma > 1 \quad \text{and} \quad \gamma = 1, 2\beta < \sigma^2 \quad \text{and} \quad \gamma \in (1/2, 1), \beta \leq 0 \quad \text{and} \quad \gamma \in (0, 1/2], \beta < 0.$$

Exercise 5. Exercise 6.10 in Klebaner's book.

Solution. See the solution on page 397 in Klebaner's book.

Exercise 6. Let X be a standard normal distributed random variable. Show how X can be made to have any given probability density function $f : \mathbb{R} \rightarrow [0, \infty)$ by means of a change of probability measure. Also, if X has probability density function $f : \mathbb{R} \rightarrow [0, \infty)$, is it possible to make X have standard normal distributed by a change of probability measure?

Solution. Clearly X has probability density function f under the probability measure

$$\mathbf{Q}(A) = \int_A f(X) \sqrt{2\pi} e^{X^2/2} d\mathbf{P} \quad \text{for } A \in \mathcal{F},$$

as this gives

$$\begin{aligned} \mathbf{Q}\{X \in B\} &= \mathbf{E}_{\mathbf{Q}}\{I_{\{X \in B\}}\} \\ &= \mathbf{E}_{\mathbf{P}}\{I_{\{X \in B\}} f(X) \sqrt{2\pi} e^{X^2/2}\} \\ &= \int_{\mathbb{R}} I_B(x) f(x) \sqrt{2\pi} e^{x^2/2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= \int_B f(x) dx \quad \text{for } B \subseteq \mathbb{R}. \end{aligned}$$

If X instead has a strictly positive probability density function $f : \mathbb{R} \rightarrow (0, \infty)$ from the beginning, then X is standard normal distributed under the probability measure

$$\mathbf{Q}(A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-X^2/2} \frac{1}{f(X)} d\mathbf{P} \quad \text{for } A \in \mathcal{F},$$

as this gives

$$\begin{aligned} \mathbf{Q}\{X \in B\} &= \mathbf{E}_{\mathbf{Q}}\{I_{\{X \in B\}}\} \\ &= \mathbf{E}_{\mathbf{P}}\left\{I_{\{X \in B\}} \frac{1}{\sqrt{2\pi}} e^{-X^2/2} \frac{1}{f(X)}\right\} \\ &= \int_{\mathbb{R}} I_B(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{f(x)} f(x) dx \\ &= \int_B \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad \text{for } B \subseteq \mathbb{R}. \end{aligned}$$

If f is not strictly positive, then it is not possible to make X standard normal distributed by means of this approach, as we then have

$$\begin{aligned}
\mathbf{Q}\{\Omega\} &= \mathbf{E}_{\mathbf{Q}}\{1\} \\
&= \mathbf{E}_{\mathbf{P}}\left\{\frac{1}{\sqrt{2\pi}} e^{-X^2/2} \frac{1}{f(X)}\right\} \\
&= \int_{\{x \in \mathbb{R}: f(x) > 0\}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{f(x)} f(x) dx \\
&= \int_{\{x \in \mathbb{R}: f(x) > 0\}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&< 1,
\end{aligned}$$

so that \mathbf{Q} is no longer a probability measure.