TMS 165/MSA350 Stochastic Calculus Part I Fall 2011

Exercise session 6

Exercise 1. Exercise 1 in Stig Larsson's lecture notes "Numerical Methods for Stochastic ODEs".

Solution. The task is to prove the estimate (8) given the conditions (1)-(5) and with Grönwall's lemma and Doob's inequality at our disposal. To land the proof we will assume that $X \in E_T$, so that the Itô integral involved is a square-integrable martingale: Without this additional assumption (which can be waived at the cost of involving the much deeper Burkholder-Davis-Gundy inequality on page 201 in Klebaner's book for local martingales) we neither know that the Itô integral involved is a martingale or that the isometry formula can be used, which is not what is really meant:) Now the proof is just a version of that of Theorem 3, as we have

$$\frac{1}{3}\mathbf{E}\Big\{\sup_{s\in[0,t]} X(s)^2\Big\}$$

$$= \frac{1}{3}\mathbf{E}\Big\{\sup_{s\in[0,t]} \Big(X_0 + \int_0^s \mu(X(r),r)\,dr + \int_0^s \sigma(X(r),r)\,dB(r)\Big)^2\Big\}$$

$$\leq \mathbf{E}\{X_0^2\} + \mathbf{E}\Big\{\sup_{s\in[0,t]}\Big(\int_0^s L\,(1+|X(r)|)\,dr\Big)^2\Big\} + \mathbf{E}\Big\{\sup_{s\in[0,t]}\Big(\int_0^s \sigma(X(r),r)\,dB(r)\Big)^2\Big\}$$

$$\leq \mathbf{E}\{X_0^2\} + \mathbf{E}\Big\{\int_0^t L^2\,(1+|X(r)|)^2\,dr \int_0^t 1^2\,dr\Big\} + \Big(\frac{2}{2-1}\Big)^2 \mathbf{E}\Big\{\Big(\int_0^t \sigma(X(r),r)\,dB(r)\Big)^2\Big\}$$

$$\leq \mathbf{E}\{X_0^2\} + 2\,L^2\,T^2 + 2\,L^2\,T\int_0^t \mathbf{E}\{X(r)^2\}\,dr + 4\int_0^t \mathbf{E}\{\sigma(X(r),r)^2\}\,dr$$

$$\leq \mathbf{E}\{X_0^2\} + 2\,L^2\,T^2 + 2\,L^2\,T\int_0^t \mathbf{E}\{X(r)^2\}\,dr + 8\,L^2\,T + 8\,L^2\int_0^t \mathbf{E}\{X(r)^2\}\,dr$$

$$\leq \mathbf{E}\{X_0^2\} + 2\,L^2\,T^2 + 8\,L^2\,T + (2\,L^2\,T + 8\,L^2)\int_0^t \sup_{r\in[0,s]}\mathbf{E}\{X(r)^2\}\,dr$$

$$\leq \Big(\mathbf{E}\{X_0^2\} + 2\,L^2\,T^2 + 8\,L^2\,T\Big)\exp\Big\{(2\,L^2\,T + 8\,L^2)\,t\Big\}$$

$$\leq \Big(\mathbf{E}\{X_0^2\} + 2\,L^2\,T^2 + 8\,L^2\,T\Big)\exp\Big\{(2\,L^2\,T + 8\,L^2)\,T\Big\} \quad \text{for } t\in[0,T].$$

Exercise 2. Exercise 2 in Larsson's lecture notes.

Solution. Consider the operator $E_T \ni Y \to G(Y) \in E_T$ given by

$$G(Y)(t) = X_0 + \int_0^t \mu(Y(r),r)\,dr + \int_0^t \sigma(Y(r),r)\,dB(r) \quad \text{for } t\in[0,T].$$

Note that Exercise 1 ensures that G maps E_T on E_T. Further, note that E_T is a Banach

space when equipped with the norm $\|X\| = \sup_{t\in[0,T]} \sqrt{E\{X(t)^2\}}$. By a version of the argument employed in the solution of Exercise 1 we have

$$\frac{1}{2}\,\|G(Y)-G(Z)\|^2$$

$$\leq \frac{1}{2}\,\mathbf{E}\Big\{\sup_{t\in[0,T]}(G(Y)(t)-G(Z)(t))^2\Big\}$$

$$= \frac{1}{2}\,\mathbf{E}\Big\{\sup_{t\in[0,T]}\Big(\int_0^t(\mu(Y(s),s)-\mu(Z(s),s))\,ds + \int_0^t(\sigma(Y(s),s)-\sigma(Z(s),s))\,dB(s)\Big)^2\Big\}$$

$$\leq \mathbf{E}\Big\{\sup_{t\in[0,T]}\Big(\int_0^t L\,|Y(s)-Z(s)|\,ds\Big)^2\Big\} + \mathbf{E}\Big\{\sup_{t\in[0,T]}\Big(\int_0^t(\sigma(Y(s),s)-\sigma(Z(s),s))\,dB(s)\Big)^2\Big\}$$

$$\leq \mathbf{E}\Big\{\int_0^T L^2\,|Y(s)-Z(s)|^2\,ds\int_0^T 1^2\,ds\Big\} + 4\,\mathbf{E}\Big\{\Big(\int_0^T(\sigma(Y(s),s)-\sigma(Z(s),s))\,dB(s)\Big)^2\Big\}$$

$$\leq L^2 T\int_0^T \mathbf{E}\big\{|Y(s)-Z(s)|^2\big\}\,ds + 4\int_0^T \mathbf{E}\big\{(\sigma(Y(s),s)-\sigma(Z(s),s))^2\big\}\,ds$$

$$\leq (L^2 T + 4L^2)\int_0^T \mathbf{E}\big\{|Y(s)-Z(s)|^2\big\}\,ds$$

$$\leq (L^2 T^2 + 4L^2 T)\,\|Y-Z\|^2.$$

Hence G is a contraction so that the fixed point theorem for contarctions on Banach spaces ensures that there exists a unique $X \in E_T$ such that $G(X) = X$. This in turn is of course a unique strong solution to the SDE (1).

Exercise 3. Exercise 3 in Larsson's lecture notes.

Solution. By a version of the argument employed in the solution of Exercise 1 we have

$$\frac{1}{3}\,\mathbf{E}\big\{(\hat{X}(t)-X(t))^2\big\}$$

$$= \frac{1}{3}\,\mathbf{E}\Big\{\Big(\hat{X}_0-X_0+\int_0^t(\mu(\hat{X}(s),s)-\mu(X(s),s))\,ds + \int_0^t(\sigma(\hat{X}(s),s)-\sigma(X(s),s))\,dB(s)\Big)^2\Big\}$$

$$\leq \mathbf{E}\big\{(\hat{X}_0-X_0)^2\big\} + \mathbf{E}\Big\{\Big(\int_0^t L\,|\hat{X}(s)-X(s)|\,ds\Big)^2\Big\} + \mathbf{E}\Big\{\Big(\int_0^t(\sigma(\hat{X}(s),s)-\sigma(X(s),s))\,dB(s)\Big)^2\Big\}$$

$$\leq \mathbf{E}\big\{(\hat{X}_0-X_0)^2\big\} + L^2 T\int_0^t \mathbf{E}\big\{(\hat{X}(s)-X(s))^2\big\}\,ds + L^2\int_0^t \mathbf{E}\big\{(\hat{X}(s)-X(s))^2\big\}\,ds$$

$$\leq \mathbf{E}\big\{(\hat{X}_0-X_0)^2\big\}\,\exp\big\{L^2(1+T)\,t\big\}$$

$$\leq \mathbf{E}\big\{(\hat{X}_0-X_0)^2\big\}\,\exp\big\{L^2(1+T)\,T\big\} \quad \text{for } t\in[0,T].$$

Exercise 4. Given some constants $\mu, \sigma \in \mathbb{R}$, consider the SDE

$$dX(t) = \mu\,dt + \sigma X(t)\,dB(t) \quad \text{for } t\in(0,T], \qquad X(0) = X_0.$$

(a) Show that the unique strong solution to this SDE is given by

$$X(t) = e^{\sigma B(t) - \sigma^2 t/2} \left(X_0 + \mu \int_0^t e^{-\sigma B(s) + \sigma^2 s/2} \, ds \right) \quad \text{for } t \in [0, T].$$

(b) Solve the SDE numerically for $\mu = \sigma = 1$, $T = 10$ and $X_0 = 0$ by the Euler method. Plot a sample path of the numerical solution and compare with the analytic solution.

Solution. **(a)** According to Theorem 5.4 in Klebaner's book the SDE has a unique strong solution. By application of Itô's formula Theorem 4.17 in Klebaner's book with

$$f(y, z) = y\,z, \quad Y(t) = e^{\sigma B(t) - \sigma^2 t/2} \quad \text{and} \quad Z(t) = X_0 + \mu \int_0^t e^{-\sigma B(t) + \sigma^2 s/2} \, ds$$

(cf. Example 4.25 in Klebaner's book) we further see that $Y(t)Z(t)$ is the solution as

$$d(Y(t)Z(t)) = Y(t)\,dZ(t) + Z(t)\,dY(t) + dY(t)dZ(t)$$

$$= \mu\,dt + Y(t)Z(t)\,\sigma\,dB(t) + (\mu\,dt)\,(\sigma\,dB(t))$$

$$= \mu\,dt + Y(t)Z(t)\,\sigma\,dB(t).$$

(b)

```
In[25]:= Clear[steps, mu, sigma, T, X0, dB, Xnumeric, B, IntB, Xanalytic]; steps = 10 000; mu = 1; sigma = 1;
         T = 10; X0 = 0; dB = Table[Random[NormalDistribution[0, Sqrt[T/steps]]], {i, 1, steps}];

In[35]:= For[i = 1; Xnumeric = {X0}, i ≤ steps, i++,
           AppendTo[Xnumeric, Xnumeric[[i]] + mu * T/steps + sigma * Xnumeric[[i]] * dB[[i]]]];
         ListPlot[Xnumeric, PlotJoined → True, PlotRange → {-0.51, 20.1},
           Ticks → {{{1000, ""}, {2000, "2"}, {3000, ""}, {4000, "4"}, {5000, ""},
             {6000, "6"}, {7000, ""}, {8000, "8"}, {9000, ""}, {10 000, "10"}}, Automatic}]
```
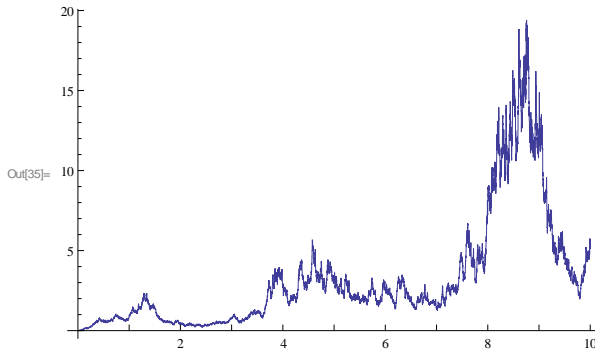


```
In[36]:= For[i = 1; Xanalytic = {X0}; B = {0}; IntB = {0}, i ≤ steps, i++, AppendTo[B, B[[i]] + dB[[i]]];
           AppendTo[IntB, IntB[[i]] + Exp[-sigma * B[[i]]] + sigma^2 * i * (T/steps) /2] * T/steps];
           AppendTo[Xanalytic, Exp[sigma * B[[i + 1]] - sigma^2 * (i + 1) * (T/steps) /2] * (X0 + IntB[[i + 1]])]];
         ListPlot[Xanalytic, PlotJoined → True, PlotRange → {-0.51, 20.1},
           Ticks → {{{1000, ""}, {2000, "2"}, {3000, ""}, {4000, "4"}, {5000, ""},
             {6000, "6"}, {7000, ""}, {8000, "8"}, {9000, ""}, {10 000, "10"}}, Automatic}]
```
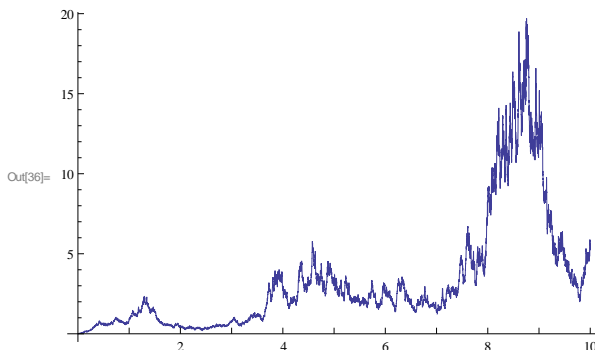
Exercise 5. Exercise 4 in Stig Larsson's lecture notes.

Solution. In order to prove (37) we note that the first equation in the proof of Theorem 6 holds with T replaced by any $s \in [t, T]$. As the expectation of the martingale on the right-hand side of that equation is zero it follows that $\mathbf{E}\{u(Z(s; x, t), s)\} = \mathbf{E}\{u(Z(t; x, t), t)\}$ for $s \in [t, T]$. Recalling that $u(Z(t; x, t), t) = u(x, t)$ since $Z(t; x, t) = x$ is the initial value of the SDE that is posted just before the proof, we arrive at $\mathbf{E}\{u(Z(s; x, t), s)\} = u(x, t)$ for $s \in [t, T]$ [which is the version of (37) we prove].

Exercise 6. Exercise 5 in Larsson's lecture notes.

Solution. This follows in the following elementary fashion from integration by parts together with the fact that the functions ϕ and ψ involved have compact support in $\mathbb{R} \times (0, T)$ [that is, they are zero outside a closed and bounded set in $\mathbb{R} \times (0, T)$]:

$$\int_0^T \int_{-\infty}^\infty (L\phi(x, t))\, \psi(x, t)\, dxdt$$

$$= \int_0^T \int_{-\infty}^\infty \left(\frac{\partial \phi(x, t)}{\partial t} + \frac{\partial}{\partial x}\left(\mu(x, t)\phi(x, t)\right) - \frac{\partial^2}{\partial x^2} \frac{\sigma(x, t)^2 \phi(x, t)}{2} \right) \psi(x, t)\, dxdt$$

$$= \int_{-\infty}^\infty \left[\phi(x, t)\psi(x, t) \right]_{t=0}^{t=T} dx - \int_{-\infty}^\infty \int_0^T \phi(x, t) \frac{\partial \psi(x, t)}{\partial t}\, dtdx$$

$$+ \int_0^T \left[\mu(x, t)\phi(x, t)\psi(x, t) \right]_{x=-\infty}^{x=\infty} dt - \int_0^T \int_{-\infty}^\infty \mu(x, t)\phi(x, t)\frac{\partial \psi(x, t)}{\partial x}\, dxdt$$

$$- \int_0^T \left[\left(\frac{\partial}{\partial x} \frac{\sigma(x, t)^2 \phi(x, t)}{2} \right) \psi(x, t) \right]_{x=-\infty}^{x=\infty} dt + \int_0^T \int_{-\infty}^\infty \left(\frac{\partial}{\partial x} \frac{\sigma(x, t)^2 \phi(x, t)}{2} \right) \frac{\partial \psi(x, t)}{\partial x}\, dxdt$$

$$= 0 - \int_{-\infty}^\infty \int_0^T \phi(x, t) \frac{\partial \psi(x, t)}{\partial t}\, dtdx$$

$$+ 0 - \int_0^T \int_{-\infty}^\infty \phi(x, t)\, \mu(x, t)\frac{\partial \psi(x, t)}{\partial x}\, dxdt$$

$$- 0 + \int_0^T \left[\frac{\sigma(x, t)^2 \phi(x, t)}{2} \frac{\partial \psi(x, t)}{\partial x} \right]_{x=-\infty}^{x=\infty} dt - \int_0^T \int_{-\infty}^\infty \frac{\sigma(x, t)^2 \phi(x, t)}{2} \frac{\partial^2 \psi(x, t)}{\partial x^2}\, dxdt$$

$$= \int_0^T \int_{-\infty}^\infty \phi(x, t)\left(-\frac{\partial \phi(x, t)}{\partial t} - \mu(x, t)\frac{\partial \psi(x, t)}{\partial x} - \frac{\sigma(x, t)^2}{2} \frac{\partial^2 \psi(x, t)}{\partial x^2} \right) dxdt$$

$$= \int_0^T \int_{-\infty}^\infty \phi(x, t)\, (L^\star \psi(x, t))\, dxdt.$$

Note that it is enough to require that one of the functions ϕ and ψ have compact support for the proof above to work: This will be important in the next exercise.

Exercise 7. Exercise 6 in Larsson's lecture notes.

4

Solution. Using (39) and (40) we shall prove that $u(x, t)$ given by (38) solves (34). [Note that Theorem 6 ensures that solutions to (34) are unique when they exist.] First note that (40) ensures that the terminal value $u(x, T)$ is $g(x)$ as required. In the language of Exercise 5 in Larsson's lecture notes we shall prove that $L^\star u(x, t) = 0$ for $u(x, t)$ given by (38). To that end we note that Exercise 5 in Larsson's lecture notes shows that for any function $\phi(x, t)$ that has compact support and satisfies $L\phi(x, t) = 0$ we have

$$
\int_0^T \int_{-\infty}^\infty \phi(x, t) \left(L^\star u(x, t) \right) dx dt = \int_{-\infty}^\infty \int_0^T \int_{-\infty}^\infty \phi(x, t) \left(L^\star p(y, T, x, t) g(y) \right) dx dt dy
$$
$$
= \int_{-\infty}^\infty \int_0^T \int_{-\infty}^\infty \left(L\phi(x, t) \right) p(y, T, x, t) g(y) \, dx dt dy
$$
$$
= 0.
$$

Assuming that the space of functions $\phi(x, t)$ with compact support that satisfy $L\phi(x, t) = 0$ is rich enough, this proves that $L^\star u(x, t) = 0$.