TMS 165/MSA350 Stochastic Calculus Part I Fall 2011

Selection from Klebaner's book for Lectures 1-11

Lecture 1, 31 August

Chapter 1

Section 1.2

Equations 1.7-1.9 [we use (1.9) instead of (1.7) although (1.9) is more general].

The notation $V_g(t)$ and the fact that this is a non-decreasing function.

Example 1.4.

Example 1.5.

Example 1.6.

Theorem 1.6.

Equation 1.13.

Theorem 1.10.

Equation 1.15.

Theorem 1.11.

Theorem 1.12.

Equation 1.17.

Section 1.3

Equations 1.18 and 1.19.

The "Particular Cases" on top of page 11.

The integration by parts formula on last row of page 12.

Lecture 2, 2 September

Chapter 2

Section 2.2

The definition of σ-field on page 28.

The definition of probability on page 29.

The definition of random variable on page 30.

Example 2.8.

The definition of σ-field generted by random variable on page 31.

Section 2.3

The first formula for $\mathbf{E}\{X\}$ of Section 2.3.

The definition of Lebesgue integral on page 33.

Equation 2.6.

The properties 1-3 of expectation on page 35.

Section 2.4

Theorems 2.16-2.18.

Lecture 3, 7 September

Sections 2.1 and 2.2

The definition of σ-field of events.

Example 2.1.

The definition of filtration.

Definition 2.1.

The definition of σ-field generated by random variable.

The definition of filtration generated by stochastic process.

Section 2.8

Definition 2.30.

Section 2.7

Everything from “General Conditional Expectation” up to and including Theorem 2.24.

Lecture 4, 9 September

Chapter 3

Section 3.1

The definition of BM - note the unspecificness of the value for $B(0)$.

Example 3.2.

Equation 3.3.

Equation 3.4 without proof.

The notation B^x and Equation 3.5.

Definition 3.1 applied to BM.

Figure 3.1.

The definition of Gaussian process.

BM is a Gaussian process.

Definition 3.2.

Theorem 3.3.

Example 3.4.

Section 3.2

Quadratic variation of BM.

Properties of Brownian paths.

Section 3.3

Begin treatment of Theorem 3.7.

Lecture 5, 14 September

Section 3.3

Finish treatment of Theorem 3.7.

Section 3.4

Definition 3.8.

Theorem 3.9.

The transition probability.

(Stopping times we will introduce later when they are needed.)

Section 3.5-3.14

This material is not included in the course.

Chapter 4

Definition 2.11.

Definition 2.12.

Theorem. (Cauchy criterion) *A sequence* $\{X_n\}_{n=1}^{\infty}$ *of random variables converges in probability to some random variable* X *if and only if*

$$\lim_{m,n\to\infty} \mathbf{P}\big\{|X_n - X_m| > \varepsilon\big\} = 0 \quad \textit{for each } \varepsilon > 0.$$

Definition 2.13.

Definition 2.14.

Theorem. (Cauchy criterion) *A sequence* $\{X_n\}_{n=1}^{\infty}$ *of random variables such that* $\mathbf{E}\{|X_n|^r\} < \infty$ *for all* n *converges in* $\mathbb{L}^r$ *to some random variable* X *if and only if*

$$\lim_{m,n\to\infty} \mathbf{E}\big\{|X_n - X_m|^r\big\} = 0.$$

Sections 4.1 - 4.2

Definition 4.2 - we use S_T to denote the class of simple adapted processes $\{X(t)\}_{t\in[0,T]}$.

The Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ for $X \in S_T$ given by Equation 4.4.

Properties of the Itô integral process for S_T - we include the martingale property Theorem 4.7 as well as continuity and adaptedness of the integral process already here.

Definition. *The class E_T consists of all adapted processes $\{X(t)\}_{t\in[0,T]}$ that satisfies*

$$\mathbf{E}\left\{\int_0^T X(t)^2\,dt\right\} < \infty.$$

Theorem. *For $X \in E_T$ there exists a sequence $\{X_n\}_{n=1}^{\infty} \subseteq S_T$ such that*

$$\lim_{n\to\infty} \mathbf{E}\left\{\int_0^T (X_n(t) - X(t))^2\,dt\right\} = 0.$$

Theorem and Definition. *For $X \in E_T$ the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ is well-defined and defined as a limit in the sense of convergence in $\mathbb{L}^2$ of $\int_0^t X_n\,dB$ as $n\to\infty$ for each $t \in [0,T]$, where $\{X_n\}_{n=1}^{\infty} \subseteq S_T$ are as in the previous theorem*

Lecture 6, 16 September

Definition. *The class E_T consists of all adapted processes $\{X(t)\}_{t\in[0,T]}$ that satisfies*

$$\mathbf{E}\bigg\{\int_0^T X(t)^2\,dt\bigg\}<\infty.$$

Theorem. *For $X\in E_T$ there exists a sequence $\{X_n\}_{n=1}^{\infty}\subseteq S_T$ such that*

$$\lim_{n\to\infty}\mathbf{E}\bigg\{\int_0^T (X_n(t)-X(t))^2\,dt\bigg\}=0.$$

Proof when X is continuous[1]. Given $X\in E_T$ and $\varepsilon>0$ we need to prove that

$$\mathbf{E}\bigg\{\int_0^T (Y(t)-X(t))^2\,dt\bigg\}\leq\varepsilon\quad\text{for some } Y\in S_T.$$

To that end let

$$X^{(N)}(t)=\begin{cases}-N & \text{if} \quad X(t)<-N\\ X(t) & \text{if} \quad |X(t)|\leq N\\ N & \text{if} \quad X(t)>N\end{cases}.$$

Since $X^{(N)}(t)-X(t)\to 0$ as $N\to\infty$ with $(X^{(N)}(t)-X(t))^2\leq X(t)^2$ we then have

$$\mathbf{E}\bigg\{\int_0^T \big(X^{(N)}(t)-X(t)\big)^2\,dt\bigg\}\to 0\quad\text{as } N\to\infty$$

(by dominated convergence Theorem 2.18 in Klebaner's book as $X\in E_T$). Using the elementary inequality $(x+y)^2\leq 2\,x^2+2\,y^2$ it follows that it is enough to prove that given $X\in E_T$, $\varepsilon>0$ and $N\in\mathbb{N}$ we have

$$\mathbf{E}\bigg\{\int_0^T \big(Y(t)-X^{(N)}(t)\big)^2\,dt\bigg\}\leq\varepsilon\quad\text{for some } Y\in S_T.$$

But as $X^{(N)}$ is uniformly continuous over $[0,T]$ (since X is uniformly continuous over $[0,T]$) we have that $Z^{(n)}\in S_T$ given by

$$Z^{(n)}(t)=I_{\{0\}}(t)\,X^{(N)}(0)+\sum_{i=0}^{n-1} I_{(t_i,t_{i+1}]}(t)\,X^{(N)}(t_i)\quad\text{for } t\in[0,T]$$

(where $0=t_0<t_1<\ldots<t_n=T$ as usual) satisfies

$$\sup_{t\in[0,T]}\big|Z^{(n)}(t)-X^{(N)}(t)\big|\leq\sup_{s,t\in[0,T],\,|s-t|\leq\max\limits_{1\leq i\leq n} t_i-t_{i-1}}\big|X^{(N)}(s)-X^{(N)}(t)\big|\to 0$$

[1]The proof for a general not necessarily continuous X is exceptionally difficult and is not really required by us as we will later restrict ourselves to continuous X'es only

as $\max_{1\le i\le n} t_i - t_{i-1} \downarrow 0$. In particular $Z^{(n)}(t) - X^{(N)}(t) \to 0$ as $\max_{1\le i\le n} t_i - t_{i-1} \downarrow 0$ with $(Z^{(n)}(t) - X^{(N)}(t))^2 \le 4\,N^2$, so that

$$\mathbf{E}\bigg\{\int_0^T \big(Z^{(n)}(t) - X^{(N)}(t)\big)^2\,dt\bigg\} \to 0 \quad \text{as} \quad \max_{1\le i\le n} t_i - t_{i-1} \downarrow 0$$

(by dominated convergence Theorem 2.18 in Klebaner's book). Hence we may pick $Y = Z^{(n)}$ for n large enough to make $\max_{1\le i\le n} t_i - t_{i-1}$ small enough. □

Theorem and Definition. *For $X \in E_T$ the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ is well-defined and defined as a limit in the sense of convergence in $\mathbb{L}^2$ of $\int_0^t X_n\,dB$ as $n \to \infty$ for each $t \in [0,T]$, where $\{X_n\}_{n=1}^\infty \subseteq S_T$ are as in the previous theorem.*

Proof. We have to show that $\{\int_0^t X_n\,dB\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{L}^2$. But this follows from the isometry property for the Itô integral for S_T as

$$\begin{aligned}
&\mathbf{E}\bigg\{\bigg(\int_0^t X_n\,dB - \int_0^t X_m\,dB\bigg)^2\bigg\} \\
&\qquad = \mathbf{E}\bigg\{\bigg(\int_0^t (X_n - X_m)\,dB\bigg)^2\bigg\} \\
&\qquad = \mathbf{E}\bigg\{\int_0^t (X_n(t) - X_m(t))^2\,dt\bigg\} \\
&\qquad \le 2\,\mathbf{E}\bigg\{\int_0^t (X_n(t) - X(t))^2\,dt\bigg\} + 2\,\mathbf{E}\bigg\{\int_0^t (X(t) - X_m(t))^2\,dt\bigg\} \\
&\qquad \to 0 \quad \text{as} \quad m,n \to \infty.
\end{aligned}$$

We must also show that if also $\{\hat{X}_n\}_{n=1}^\infty \subseteq S_T$ satisfies

$$\lim_{n\to\infty} \mathbf{E}\bigg\{\int_0^T (\hat{X}_n(t) - X(t))^2\,dt\bigg\} = 0,$$

so that $\int_0^t \hat{X}_n\,dB$ converges in $\mathbb{L}^2$ to some limit $\oint_0^t X\,dB$ as $n \to \infty$ (by what we have just proven), then $\int_0^t X\,dB = \oint_0^t X\,dB$. However, this follows from noting that

$$\begin{aligned}
&\mathbf{E}\bigg\{\bigg(\int_0^t X\,dB - \oint_0^t X\,dB\bigg)^2\bigg\} \\
&\qquad = \lim_{n\to\infty} \mathbf{E}\bigg\{\bigg(\int_0^t X_n\,dB - \int_0^t \hat{X}_n\,dB\bigg)^2\bigg\} \\
&\qquad \le 2 \lim_{n\to\infty} \mathbf{E}\bigg\{\int_0^T \big(X_n(t) - X(t)\big)^2\,dt\bigg\} + 2 \lim_{n\to\infty} \mathbf{E}\bigg\{\int_0^T \big(X(t) - \hat{X}_n(t)\big)^2\,dt\bigg\} \\
&\qquad = 0.
\end{aligned}$$

□

Properties of the Itô integral process for E_T are exactly the same as those for S_T

Definition. *The class P_T consists of all adapted processes $\{X(t)\}_{t\in[0,T]}$ that satisfies*

$$\mathbf{P}\bigg\{\int_0^T X(t)^2\,dt < \infty\bigg\} = 1.$$

Theorem. *For $X \in P_T$ we have in the sense of convergence in probability*

$$\int_0^T (X_n(t) - X(t))^2\,dt \to 0 \quad \text{as } n\to\infty \quad \text{for some sequence } \{X_n\}_{n=1}^{\infty} \subseteq E_T.$$

Theorem and Definition. *For $X \in P_T$ the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ is well-defined and defined as a limit in the sense of convergence in probability of $\int_0^t X_n\,dB$ as $n\to\infty$ for each $t\in[0,T]$, where $\{X_n\}_{n=1}^{\infty}\subseteq E_T$ are as in the previous theorem.*

Theorem. *A continuous and adapted process $\{X(t)\}_{t\in[0,T]}$ belongs to P_T and satisfies*

$$\sup_{t\in[0,T]}\bigg|\int_0^t X\,dB - \int_0^t \sum_{i=1}^n X(t_{i-1}) I_{(t_{i-1},t_i]}\,dB\bigg| \to 0 \quad \text{in probability}$$

for partitions $0=t_0<t_1<\ldots<t_n=T$ of $[0,T]$ such that $\max_{1\leq i\leq n} t_i - t_{i-1} \downarrow 0$.

Properties of the Itô integral process for P_T are the same as those for E_T, except that the properties zero-mean, martingale and isometry need no longer hold.

Example 4.2.

Example 4.5.

Theorem 4.9.

Section 4.3

Theorem 4.11.

Example 4.10.

Lecture 7, 21 September

Section 4.4

Theorem 4.13.

Example 4.12.

Example 4.13.

Section 4.5

Definition of Itô process (not same thing as Itô integral process) and stochastic differential.

Example 4.14.

Example 4.15.

Equation 4.42.

Integral wrt. Itô process.

Section 4.6

Theorem 4.16.

Example 4.20.

Example 4.23.

Theorem 4.17.

Theorem 4.18.

Example 4.26.

Lecture 8, 23 September

Chapter 5

Section 5.1

The ODE paragraph on bottom of page 123.

The SDE paragraph on top of page 126 all the way down to (not including) Example 5.3.

Example 5.5 and Example 5.3 as a special case there of.

Example 5.6.

Section 5.2

Definition of stochastic exponential on bottom of page 128.

Theorem 5.2.

Definition of stochastic logarithm on top of page 130.

Theorem 5.3. (The detailed proof is given in Exercise Session 4.)

Example 5.10.

Section 5.3

Equation 5.22 is solved by (5.30) together with (5.25), but omit the details of the verification of this fact until possibly needed later.

Obtain the solution Equation 5.32 to the Langevin equation (5.31) from Equation 5.22.

Section 5.4

Theorem 5.4. (A proof of this result is more or less included in the theory for numerical solutions of SDE covered during Lectures 12 and 13.)

Theorem 5.5.

Note that Theorems 5.4 and 5.5 only give sufficient conditions for existence and uniqueness of strong solutions to SDE, but that many a specific SDE may feature such existence and uniqueness without these sufficient conditions being satisfied as they are in fact very far from necessary.

Example 5.12.

Lecture 9, 28 September

Section 5.5

The Markov property Equation 5.41.

The transition probability Equation 5.42.

Theorem 5.6, with a motivation from the Euler scheme.

We will se a lot more on this in Lectures 12-13 on numerical mathods as well as in Lecture 14 about applications.

Section 5.6

Definition 5.8.

Definition 5.9.

Example 5.15.

Section 5.7

Theorems 5.10 and 5.11 as examples of such theorems.

The genarator L_s defined by Equation 5.50.

Definition 5.12.

Theorem 5.13 with soft proof.

Example 5.17.

Section 5.8

Definition 5.14, Theorem 5.15 and Theorem 5.16 stripped of all the technicalities.

Section 5.9

Equation 5.65.

Definition 5.17.

Theorem 5.18.

Theorem 5.19. (A detailed proof under weaker conditions is given in Exercise Session 4.)

Theorem 5.20.

Lecture 10, 30 September

Chapter 6

Section 6.1

The genarator L_t defined by Equation 5.2.

Theorem 6.2 with soft proof.

Corollary 6.4.

Example 6.2.

Section 6.2

Theorem 6.8.

Section 6.3

Definition of time homogeneous SDE in Equation 6.25.

Theorem 6.13.

Corollary 6.14.

Example 6.8.

Section 6.6

Definition 6.22.

Theorem 6.23.

Corollary 6.24.

Section 6.7

Definition 6.26.

Theorem 6.27.

Corollary 6.28.

Section 6.9

Equations 6.66-6.69.

Example 6.15.

Example 6.16.

Lecture 11, 3 October

Chapter 10

Section 10.1

Theorem 10.4.

Theorems 10.2 and 10.3 as special cases of Theorem 10.4.

Example 10.1. (A task of this type will feature on HandIn 5.)

```
Example 10.1 in Klebaner ' s book with n = 10 000 000
N[CDF[NormalDistribution[6, 1], 0]]
9.86588×10^-10
Clear[rep, x]; rep = 10 000 000;
Sum[{x = Random[NormalDistribution[0, 1]], If[x < 0, 1, 0] * Exp[6 * x - 18]}[[2]], {i, 1, rep}] / rep
9.86475×10^-10
```

Section 10.2

Everything except Example 10.3.

Section 10.3

Everything on page 275.

Equations 10.31-10.35 from the subsection "Change of Drift in Diffusions".

Theorems 10.15 and 10.16 as special cases of change of drift in diffusions.

Section 10.6

Everything from the subsection "Likelihood Ratios for Diffusions"