

## TMS 165/MSA350 Stochastic Calculus Part I Fall 2012

### Home exercises for Chapter 4 in Klebaner's book

Throughout this set of exercises  $B = \{B(t)\}_{t \geq 0}$  denotes Brownian motion.

**Task 1.** Show that a sequence  $\{X_n\}_{n=1}^\infty$  of random variables such that  $\mathbf{E}\{X_n^2\} < \infty$  for all  $n$  converges in  $\mathbb{L}^2$  to some random variable  $X$  if and only if the limit  $\lim_{m,n \rightarrow \infty} \mathbf{E}\{X_m X_n\}$  exists.

**Task 2.** Show the isometry property Equation 4.12 in Klebaner's book for the Itô integral process  $\{\int_0^t X dB\}_{t \in [0, T]}$  for  $X \in E_T$ , e.g., using that the property holds for  $X \in S_T$  [cf. Equation 4.5 in Klebaner's book].

**Task 3.** Show that for an  $X \in P_T$  we have in the sense of convergence in probability

$$\int_0^T (X_n(t) - X(t))^2 dt \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for some sequence } \{X_n\}_{n=1}^\infty \subseteq S_T,$$

and that the Itô integral process  $\{\int_0^t X dB\}_{t \in [0, T]}$  is well-defined as a limit in the sense of convergence in probability of  $\int_0^t X_n dB$  as  $n \rightarrow \infty$  for  $t \in [0, T]$ .

**Task 4.** Show that for a process  $X \in P_T$  we have

$$\mathbf{P}\left\{\int_0^T X(t)^2 dt = 0\right\} = 1 \Leftrightarrow \mathbf{P}\left\{\int_0^t X dB = 0\right\} = 1 \quad \text{for } t \in [0, T].$$

**Task 5.** Find stochastic processes  $\{X(t)\}_{t \in [0, 1]}$ ,  $\{Y(t)\}_{t \in [0, 1]}$  and  $\{Z(t)\}_{t \in [0, 1]}$  that belong to  $E_1$ ,  $P_1 \setminus E_1$  and  $P_1^c$ , respectively.

**Task 6.** Apply Itô's formula Theorem 4.17 in Klebaner's book to  $f(X(t), Y(t))$  where  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is given by  $f(x, y) = g(x)y$  for a two times continuously differentiable function  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $X = Y = B$ . Compare with what you get from applying the integration by parts formula Equation 4.57 in Klebaner's book with  $X = g(B)$  and  $Y = B$ . Derive from the comparison a new proof (without any explicit calculations other than applications of Itô's formula) of the property established in Example 4.23 in Klebaner's book that  $[g(B), B](t) = \int_0^t g'(B(s)) ds$  for  $t \geq 0$ .