# TMS 165/MSA350 Stochastic Calculus Part I Fall 2012

## Exercise session 1

**Exercise 1.** Explain why the definition of variation of a function according to Equation 1.7 in Klebaner's book is more generally valid than that according to Equation 1.9.

**Solution.** Equation 1.7 defines the variation as the supremum of a set of real numbers [namely the set of all values the sum in (1.7) can take for different partitions], and a supremum of a set of real numbers is always a well-defined quantity. Now we may consider the function $g : [0,1] \to \mathbb{R}$ given by $g(\frac{1}{2}) = 1$ and $g(x) = 0$ for $x \in [0,1] \setminus \{\frac{1}{2}\}$, which has a well-defined variation $V_g([0,1]) = 2$ according to (1.7), but which does not have a well-defined variation $V_g([0,1])$ according to (1.9).

**Exercise 2.** Prove Equations 1.16 and 1.17 in Klebaner's book for quadratic variation.

**Solution.** Clearly, (1.17) follows readily from using the definition (1.15) of quadratic variation together with a few simple algebraic manipulations. Further, we may derive (1.16) from (1.17) together with symmetry as

$$\begin{aligned}
\frac{1}{2}\big(\lfloor f+g, f+g\rfloor - \lfloor f,f\rfloor - \lfloor g,g\rfloor\big) &= \frac{1}{2}\big(\lfloor f, f+g\rfloor + \lfloor g, f+g\rfloor - \lfloor f,f\rfloor - \lfloor g,g\rfloor\big) \\
&= \frac{1}{2}\big(\lfloor f+g, f\rfloor + \lfloor f+g, g\rfloor - \lfloor f,f\rfloor - \lfloor g,g\rfloor\big) \\
&= \frac{1}{2}\big(\lfloor f,f\rfloor + \lfloor g,f\rfloor + \lfloor f,g\rfloor + \lfloor g,g\rfloor - \lfloor f,f\rfloor - \lfloor g,g\rfloor\big) \\
&= \frac{1}{2}\big(\lfloor g,f\rfloor + \lfloor f,g\rfloor\big) \\
&= \lfloor f,g\rfloor.
\end{aligned}$$

**Exercise 3.** Calculate the Stieltjes integral of a continuous function $f : [0,\infty) \to \mathbb{R}$ with respect to the function $[0,\infty) \ni x \curvearrowright g(x) = \lfloor x \rfloor \in [0,\infty)$ (i.e., the integer part of $x$) over an interval $(a,b] \subseteq [0,\infty)$.

**Solution.** Selecting integers $k, \ell \in \mathbb{N}$ such that $\lfloor a \rfloor = k$ and $\lfloor b \rfloor = \ell$, we either have $k = \ell$, in which case $k \le a < b < k+1$ and $\int_{(a,b]} f\,dg = 0$, or else $k < \ell$, in which case $k \le a < k+1 \le \ell \le b < \ell+1$ and $\int_{(a,b]} f\,dg = \sum_{n=\lfloor a \rfloor+1}^{\lfloor b \rfloor} f(n)$.

**Exercise 4.** Construct a standard normal distributed random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

**Solution.** With $\Omega = \mathbb{R}$, $\mathcal{F}$ the Borel $\sigma$-field generated by the intervals in $\mathbb{R}$ and $\mathbf{P}\{A\} = \int_A \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\, dx$ for $A \in \mathcal{F}$, the random variable $X(\omega) = \omega$ for $\omega \in \Omega$ is standard normal distributed.

**Exercise 5.** Calculate the expectation $\mathbf{E}\{X^+\}$ using the definition of the Lebesgue integral for the random variable in Exercise 4.

**Solution.** Using Example 2.9 together with page 33 in Klebaner's book we see that

$$\begin{aligned}
\mathbf{E}\{X^+\} &= \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \frac{k}{2^n}\, \mathbf{P}\big\{[\tfrac{k}{2^n}, \tfrac{k+1}{2^n})\big\} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \int_{k/2^n}^{(k+1)/2^n} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2}\, dx \\
&= \lim_{n\to\infty} \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-(k/2^n)^2/2}\, \frac{1}{2^n} \\
&= \int_0^\infty x\, \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2}\, dx \\
&= 1/\sqrt{2\pi}.
\end{aligned}$$