# TMS 165/MSA350 Stochastic Calculus Part I Fall 2012

## Exercise session 4

Througout this exercise session $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

**Exercise 1.** For two Itô processes $X = \{X(t)\}_{t\in[0,T]}$ and $Y = \{Y(t)\}_{t\in[0,T]}$ the Stratonovich integral process $\{\int_0^t X\,\partial Y\}_{t\in[0,T]}$ of $X$ wrt. $Y$ is defined as

$$\int_0^t X\,\partial Y \equiv \int_0^t X\,dY + \frac{1}{2}\,[X,Y](t) \quad \text{for } t\in[0,T]$$

(see also Section 5.9 in Klebaner's book). With this notation, show that $df(X(t)) = f'(X(t))\,\partial X(t)$ for $f$ two times continuously differentiable.

**Solution.** First we must agree on what is the exact meaning of the statement we are challanged to show, that $df(X(t)) = f'(X(t))\,\partial X(t)$. And that in turn must be that

$$f(X(t)) - f(X(0)) = \int_0^t f'(X)\,\partial X.$$

Now, by the definition of the Stratonovich integral we have

$$\int_0^t f'(X)\,\partial X = \int_0^t f'(X)\,dX + \frac{1}{2}\,[f'(X),X](t).$$

Here the arguments from Example 4.23 in Klebaner's book carry over with only obvious modifications to show that

$$[f'(X),X](t) = \int_0^t f''(X)\,d[X,X],$$

so that

$$\int_0^t f'(X)\,\partial X = \int_0^t f'(X)\,dX + \frac{1}{2}\int_0^t f''(X)\,d[X,X].$$

But the right-hand side of this in turn equals $f(X(t)) - f(X(0))$ by Itô's formula Theorem 4.16 in Klebaner's book. (Note that we only require $f$ to be two times continuously differentiable in this exercise, rather than three times continuously differentiable as is required in the corresponding Theorem 5.19 in Klebaner's book.)

**Exercise 2.** Show that for a process $X \in E_T$ the following process is a martingale

$$\left\{\left(\int_0^t X\,dB\right)^2 - \int_0^t X(s)^2\,ds\right\}_{t\in[0,T]}.$$

**Solution.** If we have proved that the above process is a martingale for $X \in S_T$, then given an $X \in E_T$, we may pick a sequence $\{X_n\}_{n=1}^{\infty} \subseteq S_T$ such that

$$\lim_{n\to\infty} \mathbf{E}\bigg\{\int_0^T (X_n(t)-X(t))^2\,dt\bigg\} = 0$$

and

$$\int_0^t X_n\,dB \to \int_0^t X\,dB \quad \text{as } n\to\infty$$

for $t\in[0,T]$ in the sense of convergence in $\mathbb{L}^2$. From this in turn we conclude by means of reapeted use of Hölder's inequality that

$$\begin{aligned}
&\mathbf{E}\bigg\{\bigg|\int_0^t X_n(s)^2\,ds - \int_0^t X(s)^2\,ds\bigg|\bigg\}\\
&= \mathbf{E}\bigg\{\bigg|\int_0^t (X_n(s)-X(s))\,(X_n(s)+X(s))\,ds\bigg|\bigg\}\\
&\le \mathbf{E}\bigg\{\sqrt{\int_0^t (X_n(s)-X(s))^2\,ds}\sqrt{\int_0^t (X_n(s)+X(s))^2\,ds}\bigg\}\\
&\le \sqrt{\mathbf{E}\bigg\{\int_0^t (X_n(s)-X(s))^2\,ds\bigg\}}\sqrt{\mathbf{E}\bigg\{\int_0^t (X_n(s)+X(s))^2\,ds\bigg\}}\\
&\le \sqrt{\mathbf{E}\bigg\{\int_0^T (X_n(s)-X(s))^2\,ds\bigg\}}\sqrt{2\,\mathbf{E}\bigg\{\int_0^T (X_n(s)-X(s))^2\,ds\bigg\}+2\,\mathbf{E}\bigg\{\int_0^T (2X(s))^2\,ds\bigg\}}\\
&\to 0 \quad \text{as } n\to\infty
\end{aligned}$$

and similarly using also the isometry property

$$\begin{aligned}
&\mathbf{E}\bigg\{\bigg|\Big(\int_0^t X_n\,dB\Big)^2 - \Big(\int_0^t X\,dB\Big)^2\bigg|\bigg\}\\
&\qquad = \mathbf{E}\bigg\{\bigg|\Big(\int_0^t X_n\,dB - \int_0^t X\,dB\Big)\Big(\int_0^t X_n\,dB + \int_0^t X\,dB\Big)\bigg|\bigg\}\\
&\qquad \le \sqrt{\mathbf{E}\bigg\{\Big(\int_0^t (X_n-X)\,dB\Big)^2\bigg\}}\sqrt{\mathbf{E}\bigg\{\Big(\int_0^t (X_n+X)\,dB\Big)^2\bigg\}}\\
&\qquad = \sqrt{\mathbf{E}\bigg\{\int_0^t (X_n(s)-X(s))^2\,ds\bigg\}}\sqrt{\mathbf{E}\bigg\{\int_0^t (X_n(s)+X(s))^2\,ds\bigg\}}\\
&\qquad \to 0 \quad \text{as } n\to\infty,
\end{aligned}$$

so that

$$\int_0^t X_n(s)^2\,ds \to \int_0^t X(s)^2\,ds \quad \text{and} \quad \Big(\int_0^t X_n\,dB\Big)^2 \to \Big(\int_0^t X\,dB\Big)^2 \quad \text{as } n\to\infty$$

for $t\in[0,T]$ in the sense of convergence in $\mathbb{L}^1$. Hence we may use Exercise 3 of Exercise Seesion 3 together with the assume proven martingale property when $X_n \in S_T$ to conclude that

$$
\begin{aligned}
\mathbf{E}\Big\{\Big(\int_0^t X\,dB\Big)^2-\int_0^t X(r)^2\,dr\,\Big|\,\mathcal{F}_s\Big\} \leftarrow \mathbf{E}\Big\{\Big(\int_0^t X_n\,dB\Big)^2-\int_0^t X_n(r)^2\,dr\,\Big|\,\mathcal{F}_s\Big\}&\\
=\Big(\int_0^s X_n\,dB\Big)^2-\int_0^s X_n(r)^2\,dr&\\
\rightarrow\Big(\int_0^s X\,dB\Big)^2-\int_0^s X(r)^2\,dr \quad \text{as } n\rightarrow\infty&
\end{aligned}
$$

for $0\leq s<t\leq T$ in the sense of convergence in $\mathbb{L}^1$, thereby establishing the requested martingale property for $X\in E_T$.

Pick a grid $0=t_0<t_1<\ldots<t_n=T$ and consider an $X\in S_T$ given by

$$
X(t)=I_{\{0\}}(t)\,\eta_0+\sum_{i=0}^{n-1} I_{(t_i,t_{i+1}]}(t)\,\xi_i \quad \text{for } t\in[0,T],
$$

where $\eta_0$ is $\mathcal{F}_0$-measurable and $\xi_i$ is $\mathcal{F}_{t_i}$-measurable for $i=0,\ldots,n-1$. Recall that

$$
\int_0^t X\,dB=\begin{cases}\displaystyle\sum_{i=0}^{m-1}\xi_i\,(B(t_{i+1})-B(t_i))+\xi_m\,(B(t)-B(t_m)) & \text{for } t\in(t_m,t_{m+1}]\\ 0 & \text{for } t=0\end{cases}.
$$

In order to prove the martingale property

$$
\mathbf{E}\Big\{\Big(\int_0^t X\,dB\Big)^2-\int_0^t X(r)^2\,dr\,\Big|\,\mathcal{F}_s\Big\}=\Big(\int_0^s X\,dB\Big)^2-\int_0^s X(r)^2\,dr
$$

for $0\leq s<t\leq T$ we may without loss of generality assume that $s=t_j$ and $t=t_k$ for some $0\leq j<k\leq n$ as the grid $0=t_0<t_1<\ldots<t_n=T$ can otherwise be enriched to accomodate $s$ and $t$ without affecting the values of

$$
\Big(\int_0^t X\,dB\Big)^2-\int_0^t X(r)^2\,dr \quad \text{and} \quad \Big(\int_0^s X\,dB\Big)^2-\int_0^s X(r)^2\,dr.
$$

Here the random variable to the right is $\mathcal{F}_s$-measurable, and therefore simple algebraic manipulations show that the martingale property to be established holds if

$$
\begin{aligned}
&\mathbf{E}\Big\{\Big(\int_0^t X\,dB\Big)^2-\Big(\int_0^s X\,dB\Big)^2-\int_s^t X(r)^2\,dr\,\Big|\,\mathcal{F}_s\Big\}\\
&\quad=\mathbf{E}\Big\{\Big(\int_s^t X\,dB\Big)^2+2\int_0^s X\,dB\int_s^t X\,dB-\int_s^t X(r)^2\,dr\,\Big|\,\mathcal{F}_s\Big\}\\
&\quad=0.
\end{aligned}
$$

That this identity holds in turn follows from the facts that

$$\mathbf{E}\Big\{\int_0^s X\,dB\int_s^t X\,dB\;\Big|\;\mathcal{F}_s\Big\} = \Big(\int_0^s X\,dB\Big)\sum_{i=j}^{k-1}\mathbf{E}\big\{\xi_i\,\mathbf{E}\big\{(B(t_{i+1})-B(t_i))\big|\mathcal{F}_{t_i}\big\}\big|\mathcal{F}_s\big\} = 0$$

and similarly

$$\begin{aligned}
&\mathbf{E}\Big\{\Big(\int_s^t X\,dB\Big)^2\Big|\;\mathcal{F}_s\Big\}\\
&\quad= \sum_{i=j}^{k-1}\mathbf{E}\big\{\xi_i^2\,\mathbf{E}\big\{(B(t_{i+1})-B(t_i))^2\big|\mathcal{F}_{t_i}\big\}\big|\mathcal{F}_s\big\}\\
&\qquad+2\sum_{j\le i_1<i_2\le k-1}\mathbf{E}\big\{\xi_{i_1}\,\xi_{i_2}\,(B(t_{i_1+1})-B(t_{i_1}))\,\mathbf{E}\big\{(B(t_{i_2+1})-B(t_{i_2}))\big|\mathcal{F}_{t_{i_2}i}\big\}\big|\mathcal{F}_s\big\}\\
&\quad= \sum_{i=j}^{k-1}\mathbf{E}\big\{\xi_i^2\,(t_{i+1}-t_i)^2\big|\mathcal{F}_s\big\}+0\\
&\quad= \mathbf{E}\Big\{\int_s^t X(r)^2\,dr\;\Big|\;\mathcal{F}_s\Big\}.
\end{aligned}$$

It is tempting to try to solve the exercise by means of applying Itô's formula, which readily gives

$$\Big(\int_0^t X\,dB\Big)^2-\int_0^t X(s)^2\,ds = 2\int_0^t\Big(\int_0^s X(r)\,dB(r)\Big)\,X(s)\,dB(s).$$

Here we know that $\int_0^s X(r)\,dB(r)$ and $X(s)$ are both square-integrable. But this only implies that $\big(\int_0^s X(r)\,dB(r)\big)\,X(s)$ is integrable (rather than square-integrable) in general, and therefore we cannot conclude that the process on the right-hand side is a martingale form what we have learned so far.

**Exercise 3.** Prove Itô's formula Theorem 4.13 in Klebaner's book.

**Solution.** We shall prove that for a two times continuously differentiable function $f$ it holds that

$$f(B(t)) = f(B(0))+\int_0^t f'(B(r))\,dB(r)+\frac{1}{2}\int_0^t f''(B(r))\,dr\quad\text{for } t>0.$$

To that end we consider partitions $0=t_0<t_1<\ldots<t_n=t$ of the interval $[0,t]$ that becomes finer and finer so that $\max_{1\le i\le n} t_i-t_{i-1}\downarrow 0$. By Taylor expansion we have

$$f(B(t))-f(B(0)) = \sum_{i=1}^{n} f(B(t_i))-f(B(t_{i-1}))$$

$$= \sum_{i=1}^{n} f'(B(t_{i-1}))\,(B(t_i)-B(t_{i-1}))$$
$$+\frac{1}{2}\sum_{i=1}^{n} f''(B(t_{i-1}))\,(B(t_i)-B(t_{i-1}))^2$$
$$+\sum_{i=1}^{n}\int_{B(t_{i-1})}^{B(t_i)} (B(t_i)-r)\,(f''(r)-f''(B(t_{i-1})))\,dr.$$

Here the first term on the right-hand side converges to $\int_0^t f'(B)\,dB$ in probability as $f(B)$ is a continuous and adapted process. Moreover, recalling that the quadratic variation of $B$ over an interval equals the length of that interval it follows that the second term on the right-hand side converges to $\frac{1}{2}\int_0^t f''(B(r))\,dr$ by means of introducing a second cruder grid $\{t_j'\}_{j=1}^m$, approximating the value of $f''(B(t_{i-1}))$ by $f''(B(t_{j-1}'))$ for an appropriate $j$, and sending first $\max_{1\leq i\leq n} t_i-t_{i-1}\downarrow 0$ and then $\max_{1\leq j\leq m} t_j'-t_{j-1}'\downarrow 0$ afterwards, as this makes it possible to replace $(B(t_i)-B(t_{i-1}))^2$ with $t_i-t_{i-1}$ in the first limit as $\max_{1\leq i\leq n} t_i-t_{i-1}\downarrow 0$ and the approximation of $f''(B(t_{i-1}))$-values by $f''(B(t_{j-1}'))$-values is accurate in the second limit as $\max_{1\leq j\leq m} t_j'-t_{j-1}'\downarrow 0$ by the continuity of $f''(B)$. Finally, the third term on the right-hand side is bounded by

$$\sup_{r,s\in[0,T],\,|r-s|\leq \max\limits_{1\leq i\leq n} t_i-t_{i-1}} |f''(B(r))-f''(B(s))| \sum_{i=1}^{n}\int_{B(t_{i-1})}^{B(t_i)} (B(t_i)-r)\,dr$$
$$= \sup_{r,s\in[0,T],\,|r-s|\leq \max\limits_{1\leq i\leq n} t_i-t_{i-1}} |f''(B(r))-f''(B(s))| \sum_{i=1}^{n}\frac{(B(t_i)-B(t_{i-1}))^2}{2}$$
$$\to 0\times\frac{t}{2}.$$

**Exercise 4.** One can prove the following important generalization of Itô's formula Theorem 4.16 in Klebaner's book: For an Itô process $\{X(t)\}_{t\in[0,T]}$ all values of which belong to an open interval $I\subseteq\mathbb{R}$ with probability 1 and a two times continuously differentiable function $f:I\to\mathbb{R}$ it holds that

$$df(X(t)) = f'(X(t))\,dX(t)+\frac{1}{2}\,f''(X(t))\,d[X,X](t)\quad \text{for }\ t\in[0,T].$$

Use this result to give a detailed proof of Theorem 5.3 in Klebaner's book.

**Solution.** Let $\{U(t)\}_{t\in[0,T]}$ be a strictly positive Itô process with probability 1. Then we may apply the above mentioned generalized Itô formula to the function $Y(t) = \log(U(t))-\log(U(0))$ to conclude that

$$dY(t) = \frac{dU(t)}{U(t)}-\frac{1}{2}\,\frac{d[U](t)}{U(t)^2},$$

so that

$$U(t)\,d\Big(\log\Big(\frac{U(t)}{U(0)}\Big)+\frac{1}{2}\int_0^t\frac{d[U](r)}{U(r)^2}\Big)=U(t)\,d\Big(Y(t)+\frac{1}{2}\int_0^t\frac{d[U](r)}{U(r)^2}\Big)=dU(t).$$

This means that the Itô process

$$\mathcal{L}(U(t))\equiv\log\Big(\frac{U(t)}{U(0)}\Big)+\frac{1}{2}\int_0^t\frac{d[U](r)}{U(r)^2}$$

has stochastic exponential $U(t)$ and therefore is the stochastic logarithm of $U(t)$. By multiplying both sides of the above equation by $1/U(t)$ we also see that $\mathcal{L}(U(t))$ obeys the equation

$$d\mathcal{L}(U(t))=\frac{1}{U(t)}\,dU(t),\quad \mathcal{L}(U(0))=0.$$

(Note that this SDE is not of diffusion type in general.)

**Exercise 5.** The filtration $\{\mathcal{F}_t\}$ that features in the construction of the Itô integral process need not necessarily be the filtration $\{\mathcal{F}_t^B\}$ generated by $B$ itself, but can more generally be as in Remark 3.1 in Klebaner's book. In particular, if $\{B_1(t)\}_{t\geq 0}$ and $\{B_2(t)\}_{t\geq 0}$ are independent Brownian motions, then we may employ the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ given by $\mathcal{F}_t=\sigma(\mathcal{F}_t^{B_1},\mathcal{F}_t^{B_1})$ for $t\geq 0$ to be able to simultaneuously consider Itô integral process (and therefore also SDE) with respect to both $B_1$ and $B_2$.

The Nobel prize awarded Black-Scholes-Merton SDE

$$dX(t)=r\,X(t)\,dt+\sigma\,X(t)\,dB(t)\quad\text{for }\ t>0,\quad X(0)=x_0,$$

for future values $\{X(t)\}_{t>0}$ of a financial asset with an uncertain rate of return might be generalized to a model that can much more accurately model real worls financial assets, such as e.g., stock prices as follows: With the notation from the previous paragraph, consider the SDE (not in general of diffusion type)

$$dX(t)=r\,X(t)\,dt+\sigma(t)\,X(t)\,dB_1(t)\quad\text{for }\ t>0,\quad X(0)=x_0,$$

where the constant so called volatility parameter $\sigma\in\mathbb{R}$ of the Black-Scholes-Merton SDE has been replaced with a random volatility process $\{\sigma(t)\}_{t\geq 0}$ that can model a market that features a time variable uncertainty for the rate of the return. Solve this more general SDE when the volatility process $\{\sigma(t)\}_{t\geq 0}$ is given by the SDE

$$d\sigma(t)=-\alpha\,\sigma(t)\,dt+\beta\,dB_2(t)\quad\text{for }\ t>0,\quad \sigma(0)=\sigma_0,$$

where $\alpha, \beta > 0$ are positive real constants (as is $r$).

**Solution.** Identifying $X$ as a stochastic exponential we get

$$X(t) = x_0 \exp\left\{ rt - \frac{1}{2}\int_0^t \sigma(s)^2\, ds + \int_0^t \sigma(s)\, dB_1(s) \right\} \quad \text{for } t \geq 0$$

(see Section 5.3 in Klebaner's book), where $\sigma$ in turn is recognized as the solution to a Langevin type SDE

$$\sigma(t) = \exp\left\{ -\int_0^t \alpha(s)\, ds \right\} \left( \sigma_0 + \beta \int_0^t \exp\left\{ \int_0^s \alpha(r)\, dr \right\} dB_2(s) \right) \quad \text{for } t \geq 0$$

(see Example 5.6 and Section 5.3 in Klebaner's book).

**Exercise 6.** Solve the SDE

$$dX(t) = \left( \sqrt{1+X(t)^2} + \frac{X(t)}{2} \right) dt + \sqrt{1+X(t)^2}\, dB(t) \quad \text{for } t > 0, \quad X(0) = 0.$$

**Solution.** First notice that all conditions of Theorem 5.4 in Klebaner's book are satisfied, so that it is clear that the SDE has a well-defined and unique solution. Now, employing divine inspiration we readily arrive at the idea to try the transformation $Y(t) = \sinh^{-1}(X(t))$. By an application of Itô's formula Theorem 4.16 in Klebaner's book we then get

$$\begin{aligned} dY(t) &= \frac{1}{\sqrt{1+X(t)^2}}\, dX(t) - \frac{X(t)}{2\,(1+X(t)^2)^{3/2}}\, d[X,X](t) \\ &= dt + \frac{X(t)}{2\sqrt{1+X(t)^2}}\, dt + dB(t) - \frac{X(t)}{2\sqrt{1+X(t)^2}}\, dt \\ &= dt + dB(t), \end{aligned}$$

with the obvious solution $Y(t) = t + B(t)$ [remembering that $Y(0) = 0$]. Hence the solution to the SDE must be $X(t) = \sinh(t + B(t))$. That this process $X$ really solves the SDE is also easy to check by means of direct calculations using Itô's formula Theorem 4.18 in Klebaner's book together with the hyperbolic unit formula.