TMS165/MSA350 Stochastic Calculus Part I

Written Exam Friday 13 January 2012 2 pm – 6 pm

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Aids: None.

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Througout this exam $B = \{B(t)\}_{t\geq 0}$ is a Brownian motion. And Good Luck to you all!

Task 1. Find the quadratic variation process $\{[B^2, B^2](t)\}_{t\geq 0}$ of squared Brownian motion $\{B^2(t)\}_{t\geq 0}$. **(5 points)**

Task 2. Give an example of a Markov process $\{X(t)\}_{t\geq 0}$ that is not a martingale. **(5 points)**

Task 3. Given an Itô process $\{X(t)\}_{t\geq 0}$, let $U_1(t) = \mathrm{e}^{X(t)-X(0)-[X,X](t)/2}$ for $t \geq 0$, so that U_1 solves the stochastic exponential SDE

$$dU(t) = U(t)\,dX(t) \quad \text{for } t \geq 0, \quad U(0) = 1.$$

Show that for any other solution $\{U_2(t)\}_{t\geq 0}$ to that SDE we must have $U_2(t) = U_1(t)$ for $t \geq 0$, for example, by means of proving that $d(U_2(t)/U_1(t)) = 0$. **(5 points)**

Task 4. Given some constants $\mu, \sigma \in \mathbb{R}$, consider the SDE

$$dX(t) = \mu\,dt + \sigma X(t)\,dB(t) \quad \text{for } t \geq 0.$$

Find $\mathbf{E}\{X(t)\}$ and $\mathbf{E}\{X(t)^2\}$ for the solution $\{X(t)\}_{t\geq 0}$ when $X(0) = 0$. **(5 points)**

Task 5. Does the SDE in Task 4 have a stationary distribution? If the answer is yes, find that stationary distribution – if the answer is no, explain why not. **(5 points)**

Task 6. Find the Milstein method for numerical solution of the SDE

$$dX(t) = \left(\sqrt{1+X(t)^2} + \frac{X(t)}{2}\right) dt + \sqrt{1+X(t)^2}\,dB(t) \quad \text{for } t \in [0,T], \quad X(0) = 0.$$

(5 points)

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Solutions to Written Exam Friday 13 January 2012

Task 1. As $d(B(t)^2) = 2\,B(t)\,dB(t) + dt$ by Itô's formula, basic rules for quadratic variations and quadratic covariations give $[B(t)^2, B(t)^2] = [\int_0^t 2\,B\,dB+t, \int_0^t 2\,B\,dB+t] = [\int_0^t 2\,B\,dB, \int_0^t 2\,B\,dB] = \int_0^t 4\,B(s)^2\,ds$.

Task 2. For example, any solution $\{X(t)\}_{t\geq 0}$ to an SDE $dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t)$ with a non-zero drift coefficient $\mu(x,t)$.

Task 3. By Itô's formula we have $d(U_2(t)/U_1(t)) = dU_2(t)/U_1(t) - U_2(t)dU_1(t)/U_1(t)^2 - dU_2(t)dU_1(t)/U_1(t)^2 + U_2(t)U_1(t)^2/U_1(t)^3 = U_2(t)dX(t)/U_1(t) - U_2(t)U_1(t)dX(t)/U_1(t)^2 - U_2(t)U_1(t)dX(t)^2/U_1(t)^2 + U_2(t)U_1(t)^2dX(t)^2/U_1(t)^3 = 0$.

Task 4. We have $\mathbf{E}\{X(t)\} = \mathbf{E}\{\mu t + \int_0^t \sigma\,X(s)\,dB(s)\} = \mu t$ and $\mathbf{E}\{X(t)^2\} = (\mu t)^2 + 2\,(\mu t)\,\mathbf{E}\{\int_0^t \sigma\,X(s)\,dB(s)\} + \mathbf{E}\{(\int_0^t \sigma\,X(s)\,dB(s))^2\} = (\mu t)^2 + \sigma^2 \int_0^t \mathbf{E}\{X(s)^2\}\,ds$, giving $\frac{d}{dt}\,\mathbf{E}\{X(t)^2\} = 2\mu^2\,t + \sigma^2\,\mathbf{E}\{X(t)^2\}$ so that (by solving this ODE) $\mathbf{E}\{X(t)^2\} = \int_0^t 2\,\mu^2\,s\,\mathrm{e}^{\sigma^2(t-s)}\,ds = 2\,\mu^2(\mathrm{e}^{\sigma^2 t} - \sigma^2 t - 1)/\sigma^4$.

Task 5. For $\mu = 0$ we see that the stationary distribution is the constant value zero. For $\mu > 0$ the stationary distribution has probability density function $\pi(x)$ given by Equation 6.69 in Klebaner's book as $\pi(x) = C\,\exp\{\int_\infty^x 2\,\mu(y)/\sigma(y)^2\,dy\}/\sigma^2(x) = C\,\mathrm{e}^{-2\mu/(\sigma^2 x)}/(\sigma^2 x^2) = 2\,\mu\,\mathrm{e}^{-2\mu/(\sigma^2 x)}/(\sigma^2 x^2)$ for $x > 0$ and $\pi(x) = 0$ otherwise, while similary for $\mu < 0$ we have $\pi(x) = 2\,\mu\,\mathrm{e}^{-2\mu/(\sigma^2 x)}/(\sigma^2 x^2)$ for $x < 0$ and $\pi(x) = 0$ otherwise.

Task 6. According to Stig's lecture notes the Milstein approximative numerical solution of the SDE is given recursively by

$$X(t_{n+1}) = X(t_n) + \mu(X(t_n))\,\Delta t_n + \sigma(X(t_n))\,\Delta B_n + \frac{\sigma(X(t_n))\sigma'(X(t_n))}{2}\,((\Delta B_n)^2 - \Delta t_n)$$

for $n = 0, \ldots, N-1$, where $0 = t_0 < t_1 < \ldots < t_N = T$, $\Delta t_n = t_{n+1} - t_n$, $\Delta B_n = B(t_{n+1}) - B(t_n)$, $\mu(x) = \sqrt{1+x^2} + x/2$ and $\sigma(x) = \sqrt{1+x^2}$.