TMS165/MSA350 Stochastic Calculus Part I

Written Exam Wednesday 3 April 2013 8.30–12.30 am

TEACHER AND JOUR: Patrik Albin, telephone 070 6945709.

AIDS: None.

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated.

GOOD LUCK!

Througout this exam $B=\{B(t)\}_{t\geq 0}$ denotes a Brownian motion while $\mathcal{F}^B=\{\mathcal{F}_t^B\}_{t\geq 0}$ denotes the filtration generated by B.

Task 1. Explain why the Riemann-Stieltjes integral $\int f\,dg$ is not well-defined when $[f,g]\neq 0$. **(5 points)**

Task 2. Show that the stochastic process $\{B(t)^4-6tB(t)^2+3t^2\}_{t\geq 0}$ is a martingale with respect to $\mathcal{F}^B$. **(5 points)**

Task 3. Show that the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ of a simple adapted (to $\mathcal{F}^B$) process $\{X(t)\}_{t\in[0,T]}$ is a martingale with respect to $\mathcal{F}^B$. **(5 points)**

Task 4. Let $f:\mathbb{R}\to\mathbb{R}$ be a two times continuously differentiable and strictly increasing function. Find a diffusion type SDE

$$dX(t)=\mu(X(t))\,dt+\sigma(X(t))\,dB(t)\quad\text{for } t>0,\quad X(0)=f(0),$$

that has solution $X(t)=f(B(t))$. **(5 points)**

Task 5. Let $\{X(t)\}_{t\geq 0}$ be a time homogeneous diffusion process that has a stationary distribution and is started according to that stationary distribution at time $t=0$. Prove that X is a stationary process, which is to say that

$$\mathbf{P}\big\{X(t_1+h)\leq x_1,\ldots,X(t_n+h)\leq x_n\big\}=\mathbf{P}\big\{X(t_1)\leq x_1,\ldots,X(t_n)\leq x_n\big\}$$

for $0<t_1<\ldots<t_n$, $h\geq 0$ and $x_1,\ldots,x_n\in\mathbb{R}$. **(5 points)**

Task 6. Given constants $\mu\in\mathbb{R}$ and $\sigma>0$, find the solution $f(x,t)$ to the PDE

$$\mu\,\frac{\partial f(x,t)}{\partial x}+\frac{\sigma^2}{2}\,\frac{\partial^2 f(x,t)}{\partial x^2}+\frac{\partial f(x,t)}{\partial t}=0\quad\text{for } t\in[0,T),\quad f(x,T)=x^2.$$ **(5 points)**

TMS165/MSA350 Stochastic Calculus Part I

Solutions to Written Exam Wednesday 3 April 2013

Task 1. When $\int_0^T f\,dg$ is well-defined the limits $\lim_{\max_{1\leq i\leq n} t_i-t_{i-1}\downarrow 0} \sum_{i=1}^n f(t_i)\,(g(t_i)-g(t_{i-1}))$ and $\lim_{\max_{1\leq i\leq n} t_i-t_{i-1}\downarrow 0} \sum_{i=1}^n f(t_{i-1})\,(g(t_i)-g(t_{i-1}))$ must coincide for grids $0=t_0<t_1<\ldots<t_{n-1}<t_n=T$ that become infinitely fine in the limit. However, when $[f,g](T)\neq 0$ these limits do not coincide as their difference is precisely $[f,g](T)$.

Task 2. By Itô's formula we have $d\big(B(t)^4-6tB(t)^2+3t^2\big) = 4\,B(t)^3\,dB(t)+6\,B(t)^2\,dt - 12\,t\,B(t)\,dB(t)-6\,t\,dt-6\,B(t)^2\,dt+6\,t\,dt = 4\,B(t)^3\,dB(t)-12\,t\,B(t)\,dB(t)$, so that $B(t)^4-6\,t\,B(t)^2+3\,t^2=\int_0^t(4\,B(s)^3-12\,s\,B(s))\,dB(s)$, which in turn is a martingale as it is an Itô integral of a mean-square integrable process.

Task 3. See the solution to Exercise 4 at

`http://www.math.chalmers.se/Stat/Grundutb/CTH/tms165/1213/Exercises/exercise3.pdf`

Task 4. By Itô's formula we have $d(f(B(t)) = f'(B(t))\,dB(t)+\frac{1}{2}\,f''(B(t))\,dt$, so that we must have $\mu(x)=\frac{1}{2}\,f''(f^{-1}(x))$ and $\sigma(x)=f'(f^{-1}(x))$.

Task 5. Writing $\pi(x)$ for the stationary probability density function and supposing that $X(0)$ has that density function, we have with obvious notation using Eq. 6.67 in Klebaner's book (see also Eq. 3.4 in Klebaner's book)

$$\begin{aligned}&\mathbf{P}\big\{X(t_1+h)\leq x_1,\ldots,X(t_n+h)\leq x_n\big\}\\ &=\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}\ldots\int_{-\infty}^{x_n}\pi(x)\,p(t_1+h,x,y_1)p(t_2-t_1,y_1,y_2)\ldots p(t_n-t_{n-1},y_{n-1},y_n)\,dy_n\ldots dy_1dx\\ &=\int_{-\infty}^{x_1}\ldots\int_{-\infty}^{x_n}\pi(y_1)\,p(t_2-t_1,y_1,y_2)\ldots p(t_n-t_{n-1},y_{n-1},y_n)\,dy_n\ldots dy_1.\end{aligned}$$

As the right-hand side of this expression does not depend on $h\geq 0$ we are done.

Task 6. The corresponding SDE is $dX(t)=\mu\,dt+\sigma\,dB(t)$ with solution $X(t)=\mu\,t+\sigma\,B(t)$, so that by the Feynman-Kac formula $f(x,t)=\mathbf{E}\{X(T)^2|X(t)=x\}=\mathbf{E}\big\{(\mu T+\sigma\,(B(T)-B(t))+\sigma\,B(t))^2\big|\,\sigma\,B(t)=x-\mu t\big\}=\mathbf{E}\big\{(\mu\,(T-t)+\sigma\,(B(T)-B(t))+x)^2\big\}$ $=(\mu\,(T-t)+x)^2+\sigma^2(T-t)$. (Of course, it is easy to check that this function f satisfies the PDE.)