

# TMS 165/MSA350 Stochastic Calculus Part I Fall 2013

## Exercise session 4

Throughout this exercise session  $B = \{B(t)\}_{t \geq 0}$  denotes Brownian motion.

**Exercise 1.** For two Itô processes  $X = \{X(t)\}_{t \in [0, T]}$  and  $Y = \{Y(t)\}_{t \in [0, T]}$  the Stratonovich integral process  $\{\int_0^t X \partial Y\}_{t \in [0, T]}$  of  $X$  wrt.  $Y$  is defined as

$$\int_0^t X \partial Y \equiv \int_0^t X dY + \frac{1}{2} [X, Y](t) \quad \text{for } t \in [0, T]$$

(see also Section 5.9 in Klebaner's book). With this notation, show that  $df(X(t)) = f'(X(t)) \partial X(t)$  for  $f$  two times continuously differentiable.

**Solution.** First we must agree on what is the exact meaning of the statement we are challenged to show, that  $df(X(t)) = f'(X(t)) \partial X(t)$ . And that in turn must be that

$$f(X(t)) - f(X(0)) = \int_0^t f'(X) \partial X.$$

Now, by the definition of the Stratonovich integral we have

$$\int_0^t f'(X) \partial X = \int_0^t f'(X) dX + \frac{1}{2} [f'(X), X](t).$$

Here the arguments from Example 4.23 in Klebaner's book carry over with only obvious modifications to show that

$$[f'(X), X](t) = \int_0^t f''(X) d[X, X],$$

so that

$$\int_0^t f'(X) \partial X = \int_0^t f'(X) dX + \frac{1}{2} \int_0^t f''(X) d[X, X].$$

But the right-hand side of this in turn equals  $f(X(t)) - f(X(0))$  by Itô's formula Theorem 4.16 in Klebaner's book. (Note that we only require  $f$  to be two times continuously differentiable in this exercise, rather than three times continuously differentiable as is required in the corresponding Theorem 5.19 in Klebaner's book.)

**Exercise 2.** Show that for a process  $X \in E_T$  the following process is a martingale

$$\left\{ \left( \int_0^t X dB \right)^2 - \int_0^t X(s)^2 ds \right\}_{t \in [0, T]}.$$

**Solution.** If we have proved that the above process is a martingale for  $X \in S_T$ , then given an  $X \in E_T$ , we may pick a sequence  $\{X_n\}_{n=1}^\infty \subseteq S_T$  such that

$$\lim_{n \rightarrow \infty} \mathbf{E} \left\{ \int_0^T (X_n(t) - X(t))^2 dt \right\} = 0$$

and

$$\int_0^t X_n dB \rightarrow \int_0^t X dB \quad \text{as } n \rightarrow \infty$$

for  $t \in [0, T]$  in the sense of convergence in  $\mathbb{L}^2$ . From this in turn we conclude by means of repeated use of Hölder's inequality that

$$\begin{aligned} & \mathbf{E} \left\{ \left| \int_0^t X_n(s)^2 ds - \int_0^t X(s)^2 ds \right| \right\} \\ &= \mathbf{E} \left\{ \left| \int_0^t (X_n(s) - X(s)) (X_n(s) + X(s)) ds \right| \right\} \\ &\leq \mathbf{E} \left\{ \sqrt{\int_0^t (X_n(s) - X(s))^2 ds} \sqrt{\int_0^t (X_n(s) + X(s))^2 ds} \right\} \\ &\leq \sqrt{\mathbf{E} \left\{ \int_0^t (X_n(s) - X(s))^2 ds \right\}} \sqrt{\mathbf{E} \left\{ \int_0^t (X_n(s) + X(s))^2 ds \right\}} \\ &\leq \sqrt{\mathbf{E} \left\{ \int_0^T (X_n(s) - X(s))^2 ds \right\}} \sqrt{2 \mathbf{E} \left\{ \int_0^T (X_n(s) - X(s))^2 ds \right\} + 2 \mathbf{E} \left\{ \int_0^T (2X(s))^2 ds \right\}} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

and similarly using also the isometry property

$$\begin{aligned} & \mathbf{E} \left\{ \left| \left( \int_0^t X_n dB \right)^2 - \left( \int_0^t X dB \right)^2 \right| \right\} \\ &= \mathbf{E} \left\{ \left| \left( \int_0^t X_n dB - \int_0^t X dB \right) \left( \int_0^t X_n dB + \int_0^t X dB \right) \right| \right\} \\ &\leq \sqrt{\mathbf{E} \left\{ \left( \int_0^t (X_n - X) dB \right)^2 \right\}} \sqrt{\mathbf{E} \left\{ \left( \int_0^t (X_n + X) dB \right)^2 \right\}} \\ &= \sqrt{\mathbf{E} \left\{ \int_0^t (X_n(s) - X(s))^2 ds \right\}} \sqrt{\mathbf{E} \left\{ \int_0^t (X_n(s) + X(s))^2 ds \right\}} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

so that

$$\int_0^t X_n(s)^2 ds \rightarrow \int_0^t X(s)^2 ds \quad \text{and} \quad \left( \int_0^t X_n dB \right)^2 \rightarrow \left( \int_0^t X dB \right)^2 \quad \text{as } n \rightarrow \infty$$

for  $t \in [0, T]$  in the sense of convergence in  $\mathbb{L}^1$ . Hence we may use Exercise 3 of Exercise Session 3 together with the assume proven martingale property when  $X_n \in S_T$  to

conclude that

$$\begin{aligned}
\mathbf{E} \left\{ \left( \int_0^t X dB \right)^2 - \int_0^t X(r)^2 dr \mid \mathcal{F}_s \right\} &\leftarrow \mathbf{E} \left\{ \left( \int_0^t X_n dB \right)^2 - \int_0^t X_n(r)^2 dr \mid \mathcal{F}_s \right\} \\
&= \left( \int_0^s X_n dB \right)^2 - \int_0^s X_n(r)^2 dr \\
&\rightarrow \left( \int_0^s X dB \right)^2 - \int_0^s X(r)^2 dr \quad \text{as } n \rightarrow \infty
\end{aligned}$$

for  $0 \leq s < t \leq T$  in the sense of convergence in  $\mathbb{L}^1$ , thereby establishing the requested martingale property for  $X \in E_T$ .

Pick a grid  $0 = t_0 < t_1 < \dots < t_n = T$  and consider an  $X \in S_T$  given by

$$X(t) = I_{\{0\}}(t)\eta_0 + \sum_{i=0}^{n-1} I_{(t_i, t_{i+1}]}(t)\xi_i \quad \text{for } t \in [0, T],$$

where  $\eta_0$  is  $\mathcal{F}_0$ -measurable and  $\xi_i$  is  $\mathcal{F}_{t_i}$ -measurable for  $i = 0, \dots, n-1$ . Recall that

$$\int_0^t X dB = \begin{cases} \sum_{i=0}^{m-1} \xi_i (B(t_{i+1}) - B(t_i)) + \xi_m (B(t) - B(t_m)) & \text{for } t \in (t_m, t_{m+1}] \\ 0 & \text{for } t = 0 \end{cases}.$$

In order to prove the martingale property

$$\mathbf{E} \left\{ \left( \int_0^t X dB \right)^2 - \int_0^t X(r)^2 dr \mid \mathcal{F}_s \right\} = \left( \int_0^s X dB \right)^2 - \int_0^s X(r)^2 dr$$

for  $0 \leq s < t \leq T$  we may without loss of generality assume that  $s = t_j$  and  $t = t_k$  for some  $0 \leq j < k \leq n$  as the grid  $0 = t_0 < t_1 < \dots < t_n = T$  can otherwise be enriched to accomodate  $s$  and  $t$  without affecting the values of

$$\left( \int_0^t X dB \right)^2 - \int_0^t X(r)^2 dr \quad \text{and} \quad \left( \int_0^s X dB \right)^2 - \int_0^s X(r)^2 dr.$$

Here the random variable to the right is  $\mathcal{F}_s$ -measurable, and therefore simple algebraic manipulations show that the martingale property to be established holds if

$$\begin{aligned}
&\mathbf{E} \left\{ \left( \int_0^t X dB \right)^2 - \left( \int_0^s X dB \right)^2 - \int_s^t X(r)^2 dr \mid \mathcal{F}_s \right\} \\
&= \mathbf{E} \left\{ \left( \int_s^t X dB \right)^2 + 2 \int_0^s X dB \int_s^t X dB - \int_s^t X(r)^2 dr \mid \mathcal{F}_s \right\} \\
&= 0.
\end{aligned}$$

That this identity holds in turn follows from the facts that

$$\mathbf{E}\left\{\int_0^s X dB \int_s^t X dB \mid \mathcal{F}_s\right\} = \left(\int_0^s X dB\right) \sum_{i=j}^{k-1} \mathbf{E}\{\xi_i \mathbf{E}\{(B(t_{i+1}) - B(t_i)) \mid \mathcal{F}_{t_i}\} \mid \mathcal{F}_s\} = 0$$

and similarly

$$\begin{aligned} & \mathbf{E}\left\{\left(\int_s^t X dB\right)^2 \mid \mathcal{F}_s\right\} \\ &= \sum_{i=j}^{k-1} \mathbf{E}\{\xi_i^2 \mathbf{E}\{(B(t_{i+1}) - B(t_i))^2 \mid \mathcal{F}_{t_i}\} \mid \mathcal{F}_s\} \\ & \quad + 2 \sum_{j \leq i_1 < i_2 \leq k-1} \mathbf{E}\{\xi_{i_1} \xi_{i_2} (B(t_{i_1+1}) - B(t_{i_1})) \mathbf{E}\{(B(t_{i_2+1}) - B(t_{i_2})) \mid \mathcal{F}_{t_{i_2}}\} \mid \mathcal{F}_s\} \\ &= \sum_{i=j}^{k-1} \mathbf{E}\{\xi_i^2 (t_{i+1} - t_i)^2 \mid \mathcal{F}_s\} + 0 \\ &= \mathbf{E}\left\{\int_s^t X(r)^2 dr \mid \mathcal{F}_s\right\}. \end{aligned}$$

It is tempting to try to solve the exercise by means of applying Itô's formula, which readily gives

$$\left(\int_0^t X dB\right)^2 - \int_0^t X(s)^2 ds = 2 \int_0^t \left(\int_0^s X(r) dB(r)\right) X(s) dB(s).$$

Here we know that  $\int_0^s X(r) dB(r)$  and  $X(s)$  are both square-integrable. But this only implies that  $(\int_0^s X(r) dB(r)) X(s)$  is integrable (rather than square-integrable) in general, and therefore we cannot conclude that the process on the right-hand side is a martingale form what we have learned so far.

**Exercise 3.** Prove Itô's formula Theorem 4.13 in Klebaner's book.

**Solution.** We shall prove that for a two times continuously differentiable function  $f$  it holds that

$$f(B(t)) = f(B(0)) + \int_0^t f'(B(r)) dB(r) + \frac{1}{2} \int_0^t f''(B(r)) dr \quad \text{for } t > 0.$$

To that end we consider partitions  $0 = t_0 < t_1 < \dots < t_n = t$  of the interval  $[0, t]$  that becomes finer and finer so that  $\max_{1 \leq i \leq n} t_i - t_{i-1} \downarrow 0$ . By Taylor expansion we have

$$f(B(t)) - f(B(0)) = \sum_{i=1}^n f(B(t_i)) - f(B(t_{i-1}))$$

$$\begin{aligned}
&= \sum_{i=1}^n f'(B(t_{i-1})) (B(t_i) - B(t_{i-1})) \\
&\quad + \frac{1}{2} \sum_{i=1}^n f''(B(t_{i-1})) (B(t_i) - B(t_{i-1}))^2 \\
&\quad + \sum_{i=1}^n \int_{B(t_{i-1})}^{B(t_i)} (B(t_i) - r) (f''(r) - f''(B(t_{i-1}))) dr.
\end{aligned}$$

Here the first term on the right-hand side converges to  $\int_0^t f'(B) dB$  in probability as  $f(B)$  is a continuous and adapted process. Moreover, recalling that the quadratic variation of  $B$  over an interval equals the length of that interval it follows that the second term on the right-hand side converges to  $\frac{1}{2} \int_0^t f''(B(r)) dr$  by means of introducing a second cruder grid  $\{t'_j\}_{j=1}^m$ , approximating the value of  $f''(B(t_{i-1}))$  by  $f''(B(t'_{j-1}))$  for an appropriate  $j$ , and sending first  $\max_{1 \leq i \leq n} t_i - t_{i-1} \downarrow 0$  and then  $\max_{1 \leq j \leq m} t'_j - t'_{j-1} \downarrow 0$  afterwards, as this makes it possible to replace  $(B(t_i) - B(t_{i-1}))^2$  with  $t_i - t_{i-1}$  in the first limit as  $\max_{1 \leq i \leq n} t_i - t_{i-1} \downarrow 0$  and the approximation of  $f''(B(t_{i-1}))$ -values by  $f''(B(t'_{j-1}))$ -values is accurate in the second limit as  $\max_{1 \leq j \leq m} t'_j - t'_{j-1} \downarrow 0$  by the continuity of  $f''(B)$ . Finally, the third term on the right-hand side is bounded by

$$\begin{aligned}
&\sup_{r,s \in [0,T], |r-s| \leq \max_{1 \leq i \leq n} t_i - t_{i-1}} |f''(B(r)) - f''(B(s))| \sum_{i=1}^n \int_{B(t_{i-1})}^{B(t_i)} (B(t_i) - r) dr \\
&= \sup_{r,s \in [0,T], |r-s| \leq \max_{1 \leq i \leq n} t_i - t_{i-1}} |f''(B(r)) - f''(B(s))| \sum_{i=1}^n \frac{(B(t_i) - B(t_{i-1}))^2}{2} \\
&\rightarrow 0 \times \frac{t}{2}.
\end{aligned}$$

**Exercise 4.** One can prove the following important generalization of Itô's formula Theorem 4.16 in Klebaner's book: For an Itô process  $\{X(t)\}_{t \in [0,T]}$  all values of which belong to an open interval  $I \subseteq \mathbb{R}$  with probability 1 and a two times continuously differentiable function  $f : I \rightarrow \mathbb{R}$  it holds that

$$df(X(t)) = f'(X(t)) dX(t) + \frac{1}{2} f''(X(t)) d[X, X](t) \quad \text{for } t \in [0, T].$$

Use this result to give a detailed proof of Theorem 5.3 in Klebaner's book.

**Solution.** Let  $\{U(t)\}_{t \in [0,T]}$  be a strictly positive Itô process with probability 1. Then we may apply the above mentioned generalized Itô formula to the function  $Y(t) = \log(U(t)) - \log(U(0))$  to conclude that

$$dY(t) = \frac{dU(t)}{U(t)} - \frac{1}{2} \frac{d[U](t)}{U(t)^2},$$

so that

$$U(t) d\left(\log\left(\frac{U(t)}{U(0)}\right) + \frac{1}{2} \int_0^t \frac{d[U](r)}{U(r)^2}\right) = U(t) d\left(Y(t) + \frac{1}{2} \int_0^t \frac{d[U](r)}{U(r)^2}\right) = dU(t).$$

This means that the Itô process

$$\mathcal{L}(U(t)) \equiv \log\left(\frac{U(t)}{U(0)}\right) + \frac{1}{2} \int_0^t \frac{d[U](r)}{U(r)^2}$$

has stochastic exponential  $U(t)$  and therefore is the stochastic logarithm of  $U(t)$ . By multiplying both sides of the above equation by  $1/U(t)$  we also see that  $\mathcal{L}(U(t))$  obeys the equation

$$d\mathcal{L}(U(t)) = \frac{1}{U(t)} dU(t), \quad \mathcal{L}(U(0)) = 0.$$

(Note that this SDE is not of diffusion type in general.)

**Exercise 5.** The filtration  $\{\mathcal{F}_t\}$  that features in the construction of the Itô integral process need not necessarily be the filtration  $\{\mathcal{F}_t^B\}$  generated by  $B$  itself, but can more generally be as in Remark 3.1 in Klebaner's book. In particular, if  $\{B_1(t)\}_{t \geq 0}$  and  $\{B_2(t)\}_{t \geq 0}$  are independent Brownian motions, then we may employ the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  given by  $\mathcal{F}_t = \sigma(\mathcal{F}_t^{B_1}, \mathcal{F}_t^{B_2})$  for  $t \geq 0$  to be able to simultaneously consider Itô integral process (and therefore also SDE) with respect to both  $B_1$  and  $B_2$ .

The Nobel prize awarded Black-Scholes-Merton SDE

$$dX(t) = r X(t) dt + \sigma X(t) dB(t) \quad \text{for } t > 0, \quad X(0) = x_0,$$

for future values  $\{X(t)\}_{t > 0}$  of a financial asset with an uncertain rate of return might be generalized to a model that can much more accurately model real world financial assets, such as e.g., stock prices as follows: With the notation from the previous paragraph, consider the SDE (not in general of diffusion type)

$$dX(t) = r X(t) dt + \sigma(t) X(t) dB_1(t) \quad \text{for } t > 0, \quad X(0) = x_0,$$

where the constant so called volatility parameter  $\sigma \in \mathbb{R}$  of the Black-Scholes-Merton SDE has been replaced with a random volatility process  $\{\sigma(t)\}_{t \geq 0}$  that can model a market that features a time variable uncertainty for the rate of the return. Solve this more general SDE when the volatility process  $\{\sigma(t)\}_{t \geq 0}$  is given by the SDE

$$d\sigma(t) = -\alpha \sigma(t) dt + \beta dB_2(t) \quad \text{for } t > 0, \quad \sigma(0) = \sigma_0,$$

where  $\alpha, \beta > 0$  are positive real constants (as is  $r$ ).

**Solution.** Identifying  $X$  as a stochastic exponential we get

$$X(t) = x_0 \exp \left\{ rt - \frac{1}{2} \int_0^t \sigma(s)^2 ds + \int_0^t \sigma(s) dB_1(s) \right\} \quad \text{for } t \geq 0$$

(see Section 5.3 in Klebaner's book), where  $\sigma$  in turn is recognized as the solution to a Langevin type SDE

$$\sigma(t) = \exp \left\{ - \int_0^t \alpha(s) ds \right\} \left( \sigma_0 + \beta \int_0^t \exp \left\{ \int_0^s \alpha(r) dr \right\} dB_2(s) \right) \quad \text{for } t \geq 0$$

(see Example 5.6 and Section 5.3 in Klebaner's book).

**Exercise 6.** Solve the SDE

$$dX(t) = \left( \sqrt{1+X(t)^2} + \frac{X(t)}{2} \right) dt + \sqrt{1+X(t)^2} dB(t) \quad \text{for } t > 0, \quad X(0) = 0.$$

**Solution.** First notice that all conditions of Theorem 5.4 in Klebaner's book are satisfied, so that it is clear that the SDE has a well-defined and unique solution. Now, employing divine inspiration we readily arrive at the idea to try the transformation  $Y(t) = \sinh^{-1}(X(t))$ . By an application of Itô's formula Theorem 4.16 in Klebaner's book we then get

$$\begin{aligned} dY(t) &= \frac{1}{\sqrt{1+X(t)^2}} dX(t) - \frac{X(t)}{2(1+X(t)^2)^{3/2}} d[X, X](t) \\ &= dt + \frac{X(t)}{2\sqrt{1+X(t)^2}} dt + dB(t) - \frac{X(t)}{2\sqrt{1+X(t)^2}} dt \\ &= dt + dB(t), \end{aligned}$$

with the obvious solution  $Y(t) = t + B(t)$  [remembering that  $Y(0) = 0$ ]. Hence the solution to the SDE must be  $X(t) = \sinh(t + B(t))$ . That this process  $X$  really solves the SDE is also easy to check by means of direct calculations using Itô's formula Theorem 4.18 in Klebaner's book together with the hyperbolic unit formula.