

TMS 165/MSA350 Stochastic Calculus Part I Fall 2013

Exercise session 6

Exercise 1. Exercise 1 in Stig Larsson's lecture notes "Numerical Methods for Stochastic ODEs".

Solution. The task is to prove the estimate (8) given the conditions (1)-(5) and with Grönwall's lemma and Doob's inequality at our disposal. To land the proof we will assume that $X \in E_T$, so that the Itô integral involved is a square-integrable martingale: Without this additional assumption (which can be waived at the cost of involving the much deeper Burkholder-Davis-Gundy inequality for local martingales in Section 7.7 of Klebaner's book) we neither know that the Itô integral involved is a martingale or that the isometry formula can be used, which is not what is really meant:) Now the proof is just a version of that of Theorem 3, as we have

$$\begin{aligned} & \frac{1}{3} \mathbf{E} \left\{ \sup_{s \in [0, t]} X(s)^2 \right\} \\ &= \frac{1}{3} \mathbf{E} \left\{ \sup_{s \in [0, t]} \left(X_0 + \int_0^s \mu(X(r), r) dr + \int_0^s \sigma(X(r), r) dB(r) \right)^2 \right\} \\ &\leq \mathbf{E}\{X_0^2\} + \mathbf{E} \left\{ \sup_{s \in [0, t]} \left(\int_0^s L(1+|X(r)|) dr \right)^2 \right\} + \mathbf{E} \left\{ \sup_{s \in [0, t]} \left(\int_0^s \sigma(X(r), r) dB(r) \right)^2 \right\} \\ &\leq \mathbf{E}\{X_0^2\} + \mathbf{E} \left\{ \int_0^t L^2(1+|X(r)|)^2 dr \int_0^t 1^2 dr \right\} + \left(\frac{2}{2-1} \right)^2 \mathbf{E} \left\{ \left(\int_0^t \sigma(X(r), r) dB(r) \right)^2 \right\} \\ &\leq \mathbf{E}\{X_0^2\} + 2L^2T^2 + 2L^2T \int_0^t \mathbf{E}\{X(r)^2\} dr + 4 \int_0^t \mathbf{E}\{\sigma(X(r), r)^2\} dr \\ &\leq \mathbf{E}\{X_0^2\} + 2L^2T^2 + 2L^2T \int_0^t \mathbf{E}\{X(r)^2\} dr + 8L^2T + 8L^2 \int_0^t \mathbf{E}\{X(r)^2\} dr \\ &\leq \mathbf{E}\{X_0^2\} + 2L^2T^2 + 8L^2T + (2L^2T + 8L^2) \int_0^t \sup_{r \in [0, s]} \mathbf{E}\{X(r)^2\} dr \\ &\leq (\mathbf{E}\{X_0^2\} + 2L^2T^2 + 8L^2T) \exp\{(2L^2T + 8L^2)t\} \\ &\leq (\mathbf{E}\{X_0^2\} + 2L^2T^2 + 8L^2T) \exp\{(2L^2T + 8L^2)T\} \quad \text{for } t \in [0, T]. \end{aligned}$$

Exercise 2. Exercise 2 in Larsson's lecture notes.

Solution. Consider the operator $E_T \ni Y \rightarrow G(Y) \in E_T$ given by

$$G(Y)(t) = X_0 + \int_0^t \mu(Y(r), r) dr + \int_0^t \sigma(Y(r), r) dB(r) \quad \text{for } t \in [0, T].$$

Note that Exercise 1 ensures that G maps E_T on E_T . Further, note that E_T is a Banach

space when equipped with the norm $\|X\| = \sup_{t \in [0, T]} \sqrt{E\{X(t)^2\}}$. By a version of the argument employed in the solution of Exercise 1 we have

$$\begin{aligned}
& \frac{1}{2} \|G(Y) - G(Z)\|^2 \\
& \leq \frac{1}{2} \mathbf{E} \left\{ \sup_{t \in [0, T]} (G(Y)(t) - G(Z)(t))^2 \right\} \\
& = \frac{1}{2} \mathbf{E} \left\{ \sup_{t \in [0, T]} \left(\int_0^t (\mu(Y(s), s) - \mu(Z(s), s)) ds + \int_0^t (\sigma(Y(s), s) - \sigma(Z(s), s)) dB(s) \right)^2 \right\} \\
& \leq \mathbf{E} \left\{ \sup_{t \in [0, T]} \left(\int_0^t L |Y(s) - Z(s)| ds \right)^2 \right\} + \mathbf{E} \left\{ \sup_{t \in [0, T]} \left(\int_0^t (\sigma(Y(s), s) - \sigma(Z(s), s)) dB(s) \right)^2 \right\} \\
& \leq \mathbf{E} \left\{ \int_0^T L^2 |Y(s) - Z(s)|^2 ds \int_0^T 1^2 ds \right\} + 4 \mathbf{E} \left\{ \left(\int_0^T (\sigma(Y(s), s) - \sigma(Z(s), s)) dB(s) \right)^2 \right\} \\
& \leq L^2 T \int_0^T \mathbf{E} \{ |Y(s) - Z(s)|^2 \} ds + 4 \int_0^T \mathbf{E} \{ (\sigma(Y(s), s) - \sigma(Z(s), s))^2 \} ds \\
& \leq (L^2 T + 4L^2) \int_0^T \mathbf{E} \{ |Y(s) - Z(s)|^2 \} ds \\
& \leq (L^2 T^2 + 4L^2 T) \|Y - Z\|^2.
\end{aligned}$$

Hence G is a contraction so that the fixed point theorem for contractions on Banach spaces ensures that there exists a unique $X \in E_T$ such that $G(X) = X$. This in turn is of course a unique strong solution to the SDE (1).

Exercise 3. Exercise 3 in Larsson's lecture notes.

Solution. By a version of the argument employed in the solution of Exercise 1 we have

$$\begin{aligned}
& \frac{1}{3} \mathbf{E} \{ (\hat{X}(t) - X(t))^2 \} \\
& = \frac{1}{3} \mathbf{E} \left\{ \left(\hat{X}_0 - X_0 + \int_0^t (\mu(\hat{X}(s), s) - \mu(X(s), s)) ds + \int_0^t (\sigma(\hat{X}(s), s) - \sigma(X(s), s)) dB(s) \right)^2 \right\} \\
& \leq \mathbf{E} \{ (\hat{X}_0 - X_0)^2 \} + \mathbf{E} \left\{ \left(\int_0^t L |\hat{X}(s) - X(s)| ds \right)^2 \right\} + \mathbf{E} \left\{ \left(\int_0^t (\sigma(\hat{X}(s), s) - \sigma(X(s), s)) dB(s) \right)^2 \right\} \\
& \leq \mathbf{E} \{ (\hat{X}_0 - X_0)^2 \} + L^2 T \int_0^t \mathbf{E} \{ (\hat{X}(s) - X(s))^2 \} ds + L^2 \int_0^t \mathbf{E} \{ (\hat{X}(s) - X(s))^2 \} ds \\
& \leq \mathbf{E} \{ (\hat{X}_0 - X_0)^2 \} \exp \{ L^2 (1+T) t \} \\
& \leq \mathbf{E} \{ (\hat{X}_0 - X_0)^2 \} \exp \{ L^2 (1+T) T \} \quad \text{for } t \in [0, T].
\end{aligned}$$

Exercise 4. Given some constants $\mu, \sigma \in \mathbb{R}$, consider the SDE

$$dX(t) = \mu dt + \sigma X(t) dB(t) \quad \text{for } t \in (0, T], \quad X(0) = X_0.$$

(a) Show that the unique strong solution to this SDE is given by

$$X(t) = e^{\sigma B(t) - \sigma^2 t/2} \left(X_0 + \mu \int_0^t e^{-\sigma B(s) + \sigma^2 s/2} ds \right) \quad \text{for } t \in [0, T].$$

(b) Solve the SDE numerically for $\mu = \sigma = 1$, $T = 10$ and $X_0 = 0$ by the Euler method. Plot a sample path of the numerical solution and compare with the analytic solution.

Solution. (a) According to Theorem 5.4 in Klebaner's book the SDE has a unique strong solution. By application of Itô's formula Theorem 4.17 in Klebaner's book with

$$f(y, z) = yz, \quad Y(t) = e^{\sigma B(t) - \sigma^2 t/2} \quad \text{and} \quad Z(t) = X_0 + \mu \int_0^t e^{-\sigma B(s) + \sigma^2 s/2} ds$$

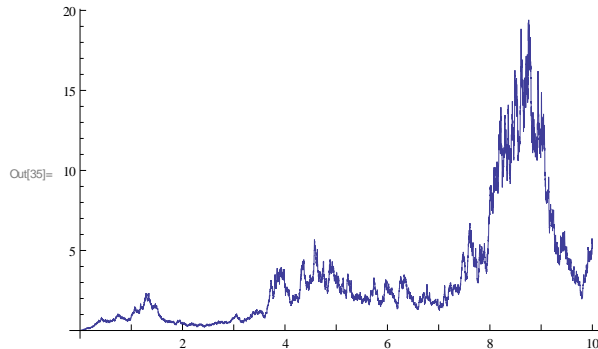
(cf. Example 4.25 in Klebaner's book) we further see that $Y(t)Z(t)$ is the solution as

$$\begin{aligned} d(Y(t)Z(t)) &= Y(t)dZ(t) + Z(t)dY(t) + dY(t)dZ(t) \\ &= \mu dt + Y(t)Z(t)\sigma dB(t) + (\mu dt)(\sigma dB(t)) \\ &= \mu dt + Y(t)Z(t)\sigma dB(t). \end{aligned}$$

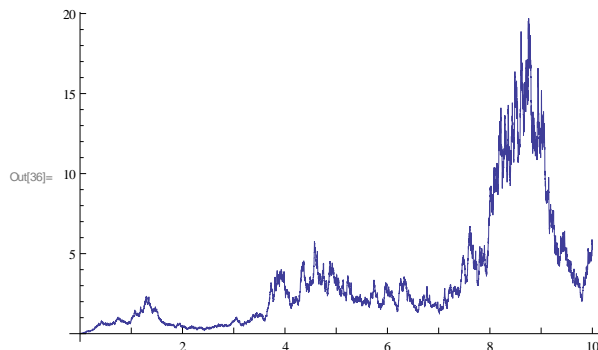
(b)

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In[25]:= Clear[steps, mu, sigma, T, X0, dB, Xnumeric, B, IntB, Xanalytic]; steps=10000; mu=1; sigma=1;
T=10; X0=0; dB=Table[Random[NormalDistribution[0, Sqrt[T/steps]]], {i, 1, steps}];
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In[35]:= For[i=1; Xnumeric={X0}, i<=steps, i++,
AppendTo[Xnumeric, Xnumeric[[i]] + mu*T/steps + sigma*Xnumeric[[i]]*dB[[i]]];
ListPlot[Xnumeric, PlotJoined->True, PlotRange->{-0.51, 20.1},
Ticks->{{1000, ""}, {2000, "2"}, {3000, ""}, {4000, "4"}, {5000, ""},
{6000, "6"}, {7000, ""}, {8000, "8"}, {9000, ""}, {10000, "10"}}, Automatic];
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In[36]:= For[i=1; Xanalytic={X0}; B={0}; IntB={0}, i<=steps, i++, AppendTo[B, B[[i]] + dB[[i]]];
AppendTo[IntB, IntB[[i]] + Exp[-sigma*B[[i]] + sigma^2*i*(T/steps)/2]*T/steps];
AppendTo[Xanalytic, Exp[sigma*B[[i+1]] - sigma^2*(i+1)*(T/steps)/2]*(X0 + IntB[[i+1]])];
ListPlot[Xanalytic, PlotJoined->True, PlotRange->{-0.51, 20.1},
Ticks->{{1000, ""}, {2000, "2"}, {3000, ""}, {4000, "4"}, {5000, ""},
{6000, "6"}, {7000, ""}, {8000, "8"}, {9000, ""}, {10000, "10"}}, Automatic];
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Exercise 5. Exercise 4 in Stig Larsson's lecture notes.

Solution. In order to prove (37) we note that the first equation in the proof of Theorem 6 holds with T replaced by any $s \in [t, T]$. As the expectation of the martingale on the right-hand side of that equation is zero it follows that $\mathbf{E}\{u(Z(s; x, t), s)\} = \mathbf{E}\{u(Z(t; x, t), t)\}$ for $s \in [t, T]$. Recalling that $u(Z(t; x, t), t) = u(x, t)$ since $Z(t; x, t) = x$ is the initial value of the SDE that is posted just before the proof, we arrive at $\mathbf{E}\{u(Z(s; x, t), s)\} = u(x, t)$ for $s \in [t, T]$ [which is the version of (37) we prove].

Exercise 6. Exercise 5 in Larsson's lecture notes.

Solution. This follows in the following elementary fashion from integration by parts together with the fact that the functions ϕ and ψ involved have compact support in $\mathbb{R} \times (0, T)$ [that is, they are zero outside a closed and bounded set in $\mathbb{R} \times (0, T)$]:

$$\begin{aligned}
& \int_0^T \int_{-\infty}^{\infty} (L\phi(x, t)) \psi(x, t) dx dt \\
&= \int_0^T \int_{-\infty}^{\infty} \left(\frac{\partial \phi(x, t)}{\partial t} + \frac{\partial}{\partial x} (\mu(x, t) \phi(x, t)) - \frac{\partial^2}{\partial x^2} \frac{\sigma(x, t)^2 \phi(x, t)}{2} \right) \psi(x, t) dx dt \\
&= \int_{-\infty}^{\infty} \left[\phi(x, t) \psi(x, t) \right]_{t=0}^{t=T} dx - \int_{-\infty}^{\infty} \int_0^T \phi(x, t) \frac{\partial \psi(x, t)}{\partial t} dt dx \\
&\quad + \int_0^T \left[\mu(x, t) \phi(x, t) \psi(x, t) \right]_{x=-\infty}^{x=\infty} dt - \int_0^T \int_{-\infty}^{\infty} \mu(x, t) \phi(x, t) \frac{\partial \psi(x, t)}{\partial x} dx dt \\
&\quad - \int_0^T \left[\left(\frac{\partial}{\partial x} \frac{\sigma(x, t)^2 \phi(x, t)}{2} \right) \psi(x, t) \right]_{x=-\infty}^{x=\infty} dt + \int_0^T \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x} \frac{\sigma(x, t)^2 \phi(x, t)}{2} \right) \frac{\partial \psi(x, t)}{\partial x} dx dt \\
&= 0 - \int_{-\infty}^{\infty} \int_0^T \phi(x, t) \frac{\partial \psi(x, t)}{\partial t} dt dx \\
&\quad + 0 - \int_0^T \int_{-\infty}^{\infty} \phi(x, t) \mu(x, t) \frac{\partial \psi(x, t)}{\partial x} dx dt \\
&\quad - 0 + \int_0^T \left[\frac{\sigma(x, t)^2 \phi(x, t)}{2} \frac{\partial \psi(x, t)}{\partial x} \right]_{x=-\infty}^{x=\infty} dt - \int_0^T \int_{-\infty}^{\infty} \frac{\sigma(x, t)^2 \phi(x, t)}{2} \frac{\partial^2 \psi(x, t)}{\partial x^2} dx dt \\
&= \int_0^T \int_{-\infty}^{\infty} \phi(x, t) \left(-\frac{\partial \phi(x, t)}{\partial t} - \mu(x, t) \frac{\partial \psi(x, t)}{\partial x} - \frac{\sigma(x, t)^2}{2} \frac{\partial^2 \psi(x, t)}{\partial x^2} \right) dx dt \\
&= \int_0^T \int_{-\infty}^{\infty} \phi(x, t) (L^* \psi(x, t)) dx dt.
\end{aligned}$$

Note that it is enough to require that one of the functions ϕ and ψ have compact support for the proof above to work: This will be important in the next exercise.

Exercise 7. Exercise 6 in Larsson's lecture notes.

Solution. Using (39) and (40) we shall prove that $u(x, t)$ given by (38) solves (34). [Note that Theorem 6 ensures that solutions to (34) are unique when they exist.] First note that (40) ensures that the terminal value $u(x, T)$ is $g(x)$ as required. In the language of Exercise 5 in Larsson's lecture notes we shall prove that $L^*u(x, t) = 0$ for $u(x, t)$ given by (38). To that end we note that Exercise 5 in Larsson's lecture notes shows that for any function $\phi(x, t)$ that has compact support and satisfies $L\phi(x, t) = 0$ we have

$$\begin{aligned} \int_0^T \int_{-\infty}^{\infty} \phi(x, t) (L^*u(x, t)) dxdt &= \int_{-\infty}^{\infty} \int_0^T \int_{-\infty}^{\infty} \phi(x, t) (L^*p(y, T, x, t)g(y)) dxdt dy \\ &= \int_{-\infty}^{\infty} \int_0^T \int_{-\infty}^{\infty} (L\phi(x, t)) p(y, T, x, t)g(y) dxdt dy \\ &= 0. \end{aligned}$$

Assuming that the space of functions $\phi(x, t)$ with compact support that satisfy $L\phi(x, t) = 0$ is rich enough, this proves that $L^*u(x, t) = 0$.