TMS 165/MSA350 Stochastic Calculus Part I Fall 2013

Home exercises for Chapter 5 in Klebaner's book

Througout this set of exercises $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. Find a diffusion type SDE that has a well-defined and unique strong solution, but that does not satisfy the conditions of Theorem 5.4 or Theorem 5.5 in Klebaner's book.

Task 2. Let $f:\mathbb{R}\to\mathbb{R}$ be a two times continuously differentiable and strictly increasing function. Find a diffusion type SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t) \quad \text{for } \ t>0, \quad X(0)=f(0),$$

that has solution $X(t) = f(B(t))$.

Task 3. Show by means of direct calculation and/or inspection (not using Theorem 5.6 in Klebaner's book) that the solution given by Equation 5.13 in Klebaner's book to the Langevin equation 5.12 in Klebaner's book is a Markov process.

Task 4. Consider a stochastic processes $\{X(t)\}_{t\geq 0}$ that is adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies

$$\mathbf{E}\{|X(t)|\} < \infty \quad \text{for } \ t\geq 0 \quad \text{ and } \quad \mathbf{E}\{X(t)\,|\,X(s)\} = X(s) \quad \text{for } \ 0\leq s\leq t.$$

Which is the most restrictive of the further requirements that X is a martingale and that X is a Markov process?

Task 5. The solution $\{X(t)\}_{t\geq 0}$ to the Langevin equation 5.12 in Klebaner's book is a Markov process with transition probability density function

$$p(t,x,y) = \frac{d}{dy}\,\mathbf{P}\{X(t+s)\leq y\,|\,X(s)=x\} = \frac{\sqrt{\alpha}}{\sqrt{\pi\,(1-\mathrm{e}^{-2\alpha t})}\,\sigma}\,\exp\biggl\{-\frac{\alpha\,(y-x\,\mathrm{e}^{-\alpha t})^2}{\sigma^2\,(1-\mathrm{e}^{-2\alpha t})}\biggr\}$$

for $s\geq 0$, $t>0$ and $x,y\in\mathbb{R}$. (This can be verified, e.g., by means of use of Equation 5.13 in Klebaner's book.) Suppose that we know the value of the parameter $\alpha>0$, but that we want to do a maximum likelihood estimation of the value of the parameter $\sigma>0$ using observations $\{x_i\}_{i=0}^n$ of the process values $\{X(i)\}_{i=0}^n$. Find that maximum likelihood estimator.