TMS 165/MSA350 Stochastic Calculus Part I Fall 2013

Home exercises for Chapters 6 and 10 in Klebaner's book

Througout this set of exercises $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. Consider the CKLS SDE from Exercise 4 of Exercise session 5

$$dX(t) = (\alpha + \beta\, X(t))\, dt + \sigma\, X(t)^{\gamma}\, dB(t) \quad \text{for } \; t > 0, \quad X(0) = X_0,$$

with parameters $\alpha, \sigma > 0$, $\beta \geq 0$ and $\gamma > 1$, and where X_0 has the stationary distribution (according to Exercise 4 of Exercise session 5) so that the solution $\{X(t)\}_{t\geq 0}$ is a stationary process (see Task 2 below). Show that

$$\mathbf{E}\bigg\{\int_0^t X(s)^{\gamma}\, dB(s)\bigg\} = C\, t \quad \text{for } \; t \geq 0,$$

for some strictly negative constant $C < 0$.

Task 2. Let $\{X(t)\}_{t\geq 0}$ be a time homogeneous diffusion process that has a stationary distribution and is started according to that stationary distribution at time $t = 0$. Prove that X is a stationary process, which is to say that

$$\mathbf{P}\big\{X(t_1{+}h) \leq x_1, \ldots, X(t_n{+}h) \leq x_n\big\} = \mathbf{P}\big\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\big\}$$

for $0 < t_1 < \ldots < t_n$ and $h > 0$.

Task 3. Find three SDE that explode, that display transcience but not explosion, and that display recurrence, respectively, but do not feature to exemplify these properties in Klebaner's book. Also, find three SDE where the issue whether the above three mentioned properties hold depends on the starting value of the SDE.

Task 4. Find a PDE that is solved by the fair price $p(x,t) = \mathbf{E}\big\{\max\{X(T){-}K, 0\}\big| X(t) = x\}$ (for a constant $K > 0$) of an European call option at time $t \in [0,T)$ for an asset price $\{X(t)\}_{t\in[0,T]}$ given by the CKLS SDE from Exercise 4 of Exercise session 5.

Task 5. Can an Ornstein-Uhlenbeck process [a solution to the Langevin equation 5.12 in Klebaner's book] become a Brownian motion by means of a change of measure? In that case, how? Otherwise, why not?

Task 6. Assume that we have observed the solution $\{X(t)\}_{t\in[0,T]}$ to the CKLS SDE in Task 1 where the γ coefficient is known. Find the estimators of the coefficients α, β and σ according to the methodolgy of Section 10.6 in Klebaner's book.