TMS 165/MSA350 Stochastic Calculus Part I Fall 2013

Home exercises for the lectures on numerical methods

Througout this set of exercises $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. It is not hard at all to find SDE for which the Euler (Euler-Maruyama) method collapses with a very high probability – one exemple would be the CKLS SDE (see Task 2 below) with a large γ coefficient. For such SDE it is usually the case that a so called implicit method works much better: With the notation of Section 1.2 in Stig Larsson's lecture notes, the so called fully implicit Euler method is given by $Y_0 = X_0$ and

$$Y_{n+1} = Y_n + \bigg(\mu(Y_{n+1}, t_{n+1}) - \Big(\sigma \frac{\partial \sigma}{\partial x}\Big)(Y_{n+1}, t_{n+1})\bigg) \int_{t_n}^{t_{n+1}} ds + \sigma(Y_{n+1}, t_{n+1}) \int_{t_n}^{t_{n+1}} dB$$

for $n \geq 0$. Explain why this method does not take the simpler form

$$Y_0 = X_0 \quad \text{and} \quad Y_{n+1} = Y_n + \mu(Y_{n+1}, t_{n+1}) \int_{t_n}^{t_{n+1}} ds + \sigma(Y_{n+1}, t_{n+1}) \int_{t_n}^{t_{n+1}} dB \quad \text{for } n \geq 0.$$

Task 2. Demonstrate by means of simulations the fact that for the CKLS SDE

$$dX(t) = (\alpha + \beta\, X(t))\, dt + \sigma\, X(t)^{\gamma}\, dB(t) \quad \text{for } t > 0, \quad X(0) = x_0,$$

with some choice of parameters $\alpha, \beta, \sigma, x_0 > 0$ and a large $\gamma > 1$ (e.g., $\gamma = 5$), the Euler method breaks down while the fully implicit Euler method works.

Task 3. Show by means of simulations of the SDE in Exercise 6 of Exercise session 4

$$dX(t) = \bigg(\sqrt{1 + X(t)^2} + \frac{X(t)}{2}\bigg)\, dt + \sqrt{1 + X(t)^2}\, dB(t) \quad \text{for } t > 0, \quad X(0) = 0,$$

that the Milstein method converges quicker than the Euler method. Are the conditions for convergence of the Euler method satisfied?

Task 4. Given constants $\mu \in \mathbb{R}$ and $\sigma > 0$, find the solution $f(x,t)$ to the PDE

$$\mu \frac{\partial f(x,t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial f(x,t)}{\partial t} = 0 \quad \text{for } t \in [0,T), \quad f(x,T) = x^2.$$

Task 5. Show that a solution $f(x,t)$ to the PDE

$$\mu(x,t) \frac{\partial f}{\partial x} + \frac{\sigma(x,t)^2}{2} \frac{\partial^2 f}{\partial x^2} + \gamma(x,t)\, f(x,t) + \frac{\partial f}{\partial t} = k(x,t) \quad \text{for } t \in [0,T), \quad f(x,T) = g(x),$$

must take the form (with obvious notation)

$$f(x,t) = \mathbf{E}\bigg\{ g(X(T))\, \mathrm{e}^{\int_t^T \gamma(X(s),s)\, ds} - \int_t^T k(X(s),s)\, \mathrm{e}^{\int_t^s \gamma(X(r),r)\, dr} ds \;\bigg|\; X(t) = x \bigg\}.$$