TMS 165/MSA350 Stochastic Calculus Part I Fall 2010

Written exam Thursday 13 January 2011 8.30 am - 1.30 pm

TEACHER AND JOUR: Patrik Albin.

AIDS: None.

GRADES: 12000 points (40%) out of the full score 30000 points to pass the exam.

MOTIVATIONS: All answers/solutions must be motivated.

Througout this exam $B = \{B(t)\}_{t\geq 0}$ is a Brownian motion.

Task 1. Find a random process (or function) with strictly positive variation but zero quadratic variation. Also, does there exist a random process (or function) with strictly positive quadratic variation but zero variation? **(5000 points)**

Task 2. Find the probability that $\mathbf{P}\{\int_0^2 B(t)\,dt > \int_0^1 B(t)\,dt\}$. **(5000 points)**

Task 3. Find the quadratic variation process of $\{\sin(B(t)\}_{t\geq 0}$. **(5000 points)**

Task 4. Find the solution $\{X(t)\}_{t\geq 0}$ to the SDE

$$dX(t) = X(t)\,\sin(B(t))\,dB(t) \quad \text{for } t>0, \quad X(0)=1. \qquad \textbf{(5000 points)}$$

Task 5. Let $X : \Omega \to \mathbb{R}$ be a standard normal distributed random variable defined on a sample space Ω with probability measure $\mathbf{P}$ and σ-field of measurable events $\mathcal{F}$. Define a new probability measure $\mathbf{Q}$ on $\mathcal{F}$ under which X is unit-mean exponential distributed. **(5000 points)**

Task 6. Write down the Milstein method for numerical solution of the Ornstein-Uhlenbeck (Langevin) SDE

$$dX(t) = -\alpha\,X(t)\,dt + \sigma\,dB(t) \quad \text{for } t>0, \quad X(0)=x_0,$$

where $\alpha,\sigma > 0$ are constants. Say as much as you can about the derivation of the method. **(5000 points)**

$\boxed{\textbf{Good Luck!}}$

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Solutions to written exam Thursday 13 January 2011

Task 1. Any continuously differentiable function $f : [0, \infty) \to \mathbb{R}$ such that $f'(0) \neq 0$ will have strictly positive variation $\int_0^t |f'(s)|\, ds$ for $t > 0$ but zero quadratic variation. Any random process (or function) that has zero variation must be constant and therefore also has zero quadratic variation.

Task 2. We have $\mathbf{P}\{\int_0^2 B(t)\, dt > \int_0^1 B(t)\, dt\} = \mathbf{P}\{\int_1^2 B(t)\, dt > 0\} = \frac{1}{2}$ as $\int_1^2 B(t)\, dt$ is a zero-mean normal distributed random variable with strictly positive variance.

Task 3. According to Itô's formula we have

$$\sin(B(t)) = \int_0^t \cos(B(s))\, dB(s) - \frac{1}{2} \int_0^t \sin(B(s))\, ds \quad \text{for } t \geq 0.$$

As the second process on the right-hand side is a continuous finite variation process the quadratic variation process of $\sin(B)$ is equal to the quadratic variation process of the first process on the right-hand side, which in turn is $\{\int_0^t \cos(B(s))^2\, ds\}_{t\geq 0}$.

Task 4. The solution X is the stochastic exponential of the martingale $\{\int_0^t \sin(B(s))\, dB(s)\}_{t\geq 0}$, which in turn is given by $\{\exp[\int_0^t \sin(B(s))\, dB(s) - \frac{1}{2}\int_0^t \sin(B(s))^2\, ds]\}_{t\geq 0}$.

Task 5. See Exercise 6 of Exercise Session 5.

Task 6. See Section 2.4 of Stig Larsson's lecture notes on numerical methods.