TMS165/MSA350 Stochastic Calculus Part I Fall 2012

Written Exam Tuesday 23 October 2012 8.30 am–12.30 am

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Aids: None.

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Good Luck!

Througout this exam $B = \{B(t)\}_{t \geq 0}$ denotes a Brownian motion.

Task 1. Find the quadratic variation process of $\sin(B(t)) + \cos(B(t))$. **(5 points)**

Task 2. If $X(t)$ is the stochastic exponential of $B(t)$, what process is then $X(t)^2$ the stochastic exponential of? **(5 points)**

Task 3. Show that the quadratic covariation between two independent Brownian motions $B_1(t)$ and $B_2(t)$ is zero. **(5 points)**

Task 4. Give an example of a diffusion process that has a stationary distribution together with its stationary distribution. Also, given an example of a diffusion process that does not have a stationary distribution and explain why this is so. **(5 points)**

Task 5. Explain how an Ornstein-Uhlenbeck process can be maid a Brownian motion by means of a change of probability measure. **(5 points)**

Task 6. Describe Milstein's method for numerical solution of SDE together with its derivation/motivation. **(5 points)**

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Solutions to Written Exam Tuesday 23 October 2012

Task 1. As $d(\sin(B(t))+\cos(B(t))) = \cos(B(t))\,dB(t) - \frac{1}{2}\sin(B(t))\,dt - \sin(B(t))\,dB(t) - \frac{1}{2}\cos(B(t))\,dt$ we have $d[\sin(B(t))+\cos(B(t)), \sin(B(t))+\cos(B(t))] = (d(\sin(B(t))+\cos(B(t))))^2 = (\cos(B(t))\,dB(t) - \sin(B(t))\,dB(t))^2 = (\cos(B(t)) - \sin(B(t)))^2\,dt$, giving $[\sin(B(t))+\cos(B(t)), \sin(B(t))+\cos(B(t))] = \int_0^t (\cos(B(s)) - \sin(B(s)))^2\,ds$.

Task 2. As $dX(t) = X(t)\,dB(t)$ we have that $Y(t) = X(t)^2$ satisfies $dY(t) = d(X(t)^2) = 2\,X(t)\,dX(t) + d[X,X](t) = 2\,X(t)^2\,dB(t) + X(t)^2\,dt = Y(t)\,(2\,dB(t) + dt)$, making $X(t)^2$ the stochastic exponential of $2\,B(t) + t$.

Task 3. As it is easy to check that $(B_1(t) - B_2(t))/\sqrt{2}$ is a Brownian motion (being a Gaussian process with the right mean and covariance functions), we have $[B_1, B_2](t) = \big([B_1, B_1](t) + [B_2, B_2](t) - 2\,[(B_1 - B_2)/\sqrt{2}, (B_1 - B_2)/\sqrt{2}](t)\big)/2 = (t + t - 2\,t)/2 = 0$.

Task 4. See Example 6.16 in Klebaner's book.

Task 5. According to Theorem 10.16 in Klebaner's book, if $dX(t) = -X(t)\,dt + dB(t)$ where B is a P-Brownian motion, then $X(t)$ is a Q-Brownian motion when $dQ/dP = \exp\{\int_0^T X(s)\,dB(s) - \frac{1}{2}\int_0^T X(s)^2\,ds\}$.

Task 6. See page 12-13 in Stig's lecture notes.