TMS165/MSA350 Stochastic Calculus Part I

Written Exam Friday 18 January 2013 8.30 am–12.30 am

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AIDS: None.

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated.

GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Solve the SDE

$$dX(t) = \frac{1}{2}X(t)(\ln(X(t)))^2\,dt + X(t)\ln(X(t))\,dB(t) \quad \text{for } t>0, \quad X(0)=\mathrm{e}.$$

(5 points)

Task 2. Find the quadratic covariation process $[\exp\{\mathrm{e}^{B(t)}\}, B(t)^3](t)$ for $t \geq 0$.

(5 points)

Task 3. Consider an Ornstein-Uhlembeck $\{X(t)\}_{t\geq 0}$ given by the SDE

$$dX(t) = -X(t)\,dt + dB(t) \quad \text{for } t>0, \quad X(0)=0.$$

For which two times continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ is $\{f(X(t))\}_{t\geq 0}$ a time-homogeneous diffusion process with zero drift coefficient? **(5 points)**

Task 4. We have observed a diffusion process $\{X(t)\}_{t\in[0,10]}$ which is either Brownian motion $X(t) = B(t)$ or an Ornstein-Uhlenbeck process as specified in Task 3 above. How can we used the observed data $\{X(t)\}_{t\in[0,10]}$ to determine which of the two models (/origins) for $X(t)$ is the correct one? **(5 points)**

Task 5. Solve the PDE

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f(x,t)}{\partial x^2} + \mu x \frac{\partial f(x,t)}{\partial x} + \frac{\partial f(x,t)}{\partial t} = r\,f(x,t) \quad \text{for } (x,t)\in\mathbb{R}\times[0,T], \quad f(x,T)=x^2,$$

where $\sigma^2, \mu, r, T > 0$ are constants. **(5 points)**

Task 6. Describe the Euler method for numerical solution of an SDE. Under what conditions on the coefficients of the SDE can the Euler method be expected to converge? Give a sketch of the proof of that convergence. **(5 points)**

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Solutions to Written Exam Friday 18 January 2013

Task 1. Taking $Y(t) = \ln(X(t))$ Itô's formula shows that

$$\begin{aligned} dY(t) &= \frac{dX(t)}{X(t)} - \frac{d[X,X](t)}{2\,X(t)^2} \\ &= \frac{1}{2}(\ln(X(t)))^2\,dt + \ln(X(t))\,dB(t) - \frac{1}{2}(\ln(X(t)))^2\,dt \\ &= Y(t)\,dB(t) \end{aligned}$$

with $Y(0) = 1$, so that $Y(t) = \mathrm{e}^{B(t)-t/2}$ and $X(t) = \exp\{\mathrm{e}^{B(t)-t/2}\}$.

Task 2. As $d(\exp\{\mathrm{e}^{B(t)}\}) = \exp\{\mathrm{e}^{B(t)}\}(\mathrm{e}^{B(t)}\,dB(t) + \frac{1}{2}(\mathrm{e}^{B(t)} + \mathrm{e}^{2B(t)})\,dt)$ and $d(B(t)^3) = 3\,B(t)^2\,dB(t) + 3\,B(t)\,dt$, we have

$$d([\exp\{\mathrm{e}^{B(t)}\}, B(t)^3](t)) = d(\exp\{\mathrm{e}^{B(t)}\})\,d(B(t)^3) = 3\,B(t)^2\,\mathrm{e}^{B(t)}\exp\{\mathrm{e}^{B(t)}\}\,dt,$$

so that $[\exp\{\mathrm{e}^{B(t)}\}, B(t)^3](t) = \int_0^t 3\,B(s)^2\,\mathrm{e}^{B(s)}\exp\{\mathrm{e}^{B(s)}\}\,ds$ for $t \geq 0$.

Task 3. The diffusion process

$$\begin{aligned} f(X(t)) &= f(X(0)) + \int_0^t f'(X(s))\,dX(s) + \frac{1}{2}\int_0^t f''(X(s))\,d[X,X](s) \\ &= f(0) + \int_0^t f'(X(s))\,dB(s) + \int_0^t \Big(\frac{1}{2}f''(X(s)) - f'(X(s))\,X(s)\Big)\,ds \end{aligned}$$

has zero drift if $\frac{1}{2}f''(x) - f'(x)\,x = 0$, which gives $\ln(f'(x)) = x^2 + C_1$ so that $f'(x) = C_2\,\mathrm{e}^{x^2}$ and $f(x) = C_3\int \mathrm{e}^{x^2}\,dx + C_4$ for $x \geq 0$, where $C_1, C_4 \in \mathbb{R}$ and $C_2, C_3 > 0$ are constants.

Task 4. According to Example 10.5 and Equation 10.53 in Klebaner's book we calculate the likelihood

$$\Lambda(X) = \exp\bigg\{-\int_0^{10} X(t)\,dX(t) - \frac{1}{2}\int_0^{10} X(t)^2\,dt\bigg\}.$$

If this likelihood is (significantly) bigger that 1 we conclude that $X(t)$ is an Ornrstein-Uhlenbeck process while if the likelihood is (significantly) smaller that 1 we conclude that $X(t)$ is a Brownian motion.

Task 5. See Example 6.5 in Klebaner's book.

Task 6. See Lecture 13 in Stig's lecture notes.