TMS165/MSA350 Stochastic Calculus Part I

Written Exam Tuesday 22 October 2013 8.30–12.30 am

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AIDS: None.

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated.

GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Find the stochastic exponential of the stochastic logarithm $\mathcal{E}(\mathcal{L}(X))(t)$ of a strictly positive Itô process $X(t)$. (Hint: The claim in Exercise 5.7 in Klebaner's book that $\mathcal{E}(\mathcal{L}(X))(t) = X(t)$ is <u>not</u> true in general ... .) **(5 points)**

Task 2. Solve the Stratonovich SDE

$$\partial X(t) = X(t)\,\partial B(t), \quad X(0) = 1.$$ **(5 points)**

Task 3. Let $X(t) = (1-t)\int_0^t (1-s)^{-1}\,dB(s)$ for $t \in [0,1)$. Find $dX(t)$. **(5 points)**

Task 4. Let $X(t)$ be the solution to an SDE with coefficients $\mu(x,t)$ and $\sigma(x,t)$. Find a differential equation that must hold for a function $f(x,t)$ which makes $Y(t) = f(X(t),t)$ a solution to an SDE with μ- and σ-coefficients 0 and 1, respectively. **(5 points)**

Task 5. Give the maximum likelihood estimator for the parameter μ based on observation of the solution $\{X(t)\}_{t\in[0,T]}$ of the SDE

$$dX(t) = \mu\,X(t)\,dt + \sigma\,X(t)\,dB(t), \quad X(0) = 1.$$ **(5 points)**

Task 6. Derive the Milstein method for numerical solution of SDE. **(5 points)**

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Solutions to Written Exam Tuesday 22 October 2013

Task 1. By inserting in Theorems 5.2 and 5.3 in Klebaner's book one readily obtains $\mathcal{E}(\mathcal{L}(X))(t) = X(t)/X(0)$.

Task 2. According to Theorem 5.10 in Klebaner's book the Stratonovich SDE corresponds to the Itô SDE

$$dX(t) = \frac{1}{2} X(t)\, dt + X(t)\, dB(t), \quad X(0) = 1,$$

which in turn according to Example 5.5 in Klebaner's book has solution $X(t) = \mathrm{e}^{B(t)}$.

Note that the equations $\partial X(t) = X(t)\, \partial B(t)$ and $dX(t) = X(t)\, \partial B(t)$ are identical as we have $\int_0^t \partial X(s) = \int_0^t dX(s)$ by Definition 5.17 in Klebaner's book (with $Y(t) = 1$).

Task 3. Writing $Y(t) = (1-t)$ and $Z(t) = \int_0^t (1-s)^{-1}\, dB(s)$, Eq. 4.59 in Klebaner's book gives

$$\begin{aligned} d(Y(t)Z(t)) &= Y(t)\, dZ(t) + Z(t)\, dY(t) + dY(t)\, dZ(t) \\ &= dB(t) - \left(\int_0^t \frac{dB(s)}{1-s}\right) dt - \frac{dB(t)}{1-t}\, dt \\ &= dB(t) - \left(\int_0^t \frac{dB(s)}{1-s}\right) dt. \end{aligned}$$

Task 4. By Itô's formula we have

$$dY(t) = df(X(t),t) = \left(\frac{\partial f}{\partial t} + \mu(X(t),t)\, \frac{\partial f}{\partial x} + \frac{\sigma(X(t),t)^2}{2}\, \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma(X(t),t)\, \frac{\partial f}{\partial x}\, dB(t).$$

Hence it is required that

$$\frac{\partial f(x,t)}{\partial t} + \mu(x,t)\, \frac{\partial f(x,t)}{\partial x} + \frac{\sigma(x,t)^2}{2}\, \frac{\partial^2 f(x,t)}{\partial x^2} = 0 \quad \text{and} \quad \sigma(x,t)\, \frac{\partial f(x,t)}{\partial x} = 1.$$

Task 5. By insertion in Eq. 10.52 in Klebaner's book with $\mu_1(x,t) = 0$, $\mu_2(x,t) = \mu\, x$ and $\sigma(x,t) = \sigma\, x$ we find that the likelihood is

$$\Lambda = \exp\left\{\int_0^T \frac{\mu\, X(t)}{\sigma^2 X(t)^2}\, dX(t) - \frac{1}{2} \int_0^T \frac{\mu^2 X(t)^2}{\sigma^2 X(t)^2}\, dt\right\}.$$

By solving the equation $\partial\Lambda/\partial\mu = 0$ for μ we obtain the maximum likelihood estimator $\frac{1}{T}\int_0^T X(t)^{-1}\, dX(t)$ of μ.

Task 6. The Milstein method is derived in Stig's lecture notes - check there how it is done.