TMS165/MSA350 Stochastic Calculus Part I

Written Exam Wednesday 23 April 2014 8.30–12.30 am

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Aids: None.

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Good Luck!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Show that $\{\int_0^t B(r)\,dr\}_{t\geq 0}$ is not a martingale wrt. the filtration $\mathcal{F}_t = \mathcal{F}_t^B$ generated by B. **(5 points)**

Task 2. Is it possible to find constants/coefficients $a, b, c \in \mathbb{R}$ such that $X(t) = B(t)^3 + a\,B(t)^2 + b\,B(t) + c\,t$ becomes a martingale? (You may make use of the fact that $\int_0^t B(s)\,ds$ is not a martingale – see Task 1.) **(5 points)**

Task 3. Given constants $\alpha > 0$ and $\beta \in \mathbb{R}$, solve the SDE

$$dX(t) = -(\alpha X(t) + \beta)\,dt + dB(t) \quad \text{for } t > 0, \quad X(0) = 1. \qquad \textbf{(5 points)}$$

Task 4. Find the solution $f(x,t)$ to the PDE

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t} = 0 \quad \text{for } t \in [0,T], \quad f(x,T) = x^2. \qquad \textbf{(5 points)}$$

Task 5. Let $\{N(t)\}_{t\geq 0}$ be a unit rate Poisson process that is independent of B, where both processes are assumed to be defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Put $\Lambda(t) = \mathrm{e}^{(\ln 2)N(t)-t}$ for $t \in [0,T]$ and $dQ = \Lambda(T)dP$. Show that $\{B(t)\}_{t\in[0,T]}$ is a Brownian motion on the interval $[0,T]$ under the probability measure Q. **(5 points)**

Task 6. Describe how the machinery of Itô-Taylor expansion works for a homogeneous SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t) \quad \text{for } t > 0. \qquad \textbf{(5 points)}$$

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Solutions to Written Exam Wednesday 23 April 2014

Task 1. As $\mathbf{E}\{\int_0^t B(r)\,dr\,|\,\mathcal{F}_s\} = \int_0^s B(r)\,dr + \mathbf{E}\{\int_s^t (B(r)-B(s))\,dr\} + (t-s)\,B(s) = \int_0^s B(r)\,dr + (t-s)\,B(s)$ it is clear that $\int_0^t B(r)\,dr$ is not a martingale.

Task 2. As by Itôs's formula $dX(t) = (3\,B(t)^2 + 2a\,B(t) + b)\,dB(t) + (3\,B(t) + a + c)\,dt$, where $\int_0^t (3\,B(s)^2 + 2a\,B(s) + b)\,dB(s)$ is a martingale, the answer to the query is yes if an only if $\int_0^t (3\,B(s) + a + c)\,ds$ is also a martingale. To give this process constant mean (as is required for a martingale) we have to have $a + c = 0$ so that the answer to the query is yes if an only if $a + c = 0$ and $3\int_0^t B(s)\,ds$ is a martingale. However we know that $3\int_0^t B(s)\,ds$ is not a martingale from Task 1.

Task 3. Writing $\hat{B}(t) = B(t) - \beta t$ we obtain the simpler Langevin type equation

$$dX(t) = -\alpha X(t)\,dt + d\hat{B}(t) \quad \text{for } t > 0, \quad X(0) = 1,$$

which by analogy with the Langevin equation (see Example 5.6 in Klebaner's book) has solution

$$X(t) = \mathrm{e}^{-\alpha t}\left(1 + \int_0^t \mathrm{e}^{\alpha s}\,d\hat{B}(s)\right) = \mathrm{e}^{-\alpha t} - \beta(1 - \mathrm{e}^{-\alpha t})/\alpha + \mathrm{e}^{-\alpha t}\int_0^t \mathrm{e}^{\alpha s}\,dB(s).$$

This process X in turn can easily be seen to verify the given SDE (if you have not been convinced about that already).

Task 4. This is Exercise 6.10 in Klebaner's book – see the solution on page 417 in that book.

Task 5. As B is a continuous process with quadratic variation $[B,B](t) = t$ it follows fron Levy's characterization of Brownian motion that it is enough to prove that B is a Q-martingale. To that end it is in turn by Corollary 10.11 in Klebaner's book necessary and sufficient to show that $\Lambda(t)B(t)$ is a $\mathbf{P}$-martingale wrt. the filtration $\mathcal{F}_t = \mathcal{F}_t^B \vee \mathcal{F}_t^N$ generated by B and N. To see this we calculate $\mathbf{E}\{\Lambda(t)B(t)\,|\,\mathcal{F}_s\} = \mathrm{e}^{(\ln 2)N(s)-t}\,\mathbf{E}\{\mathrm{e}^{(\ln 2)(N(t)-N(s))}\}\mathbf{E}\{B(t)\,|\,\mathcal{F}_s\} = \mathrm{e}^{(\ln 2)N(s)-t}\,\mathrm{e}^{t-s}\,B(s) = \Lambda(s)B(s)$ for $s \leq t$, since $\mathbf{E}\{\mathrm{e}^{(\ln 2)(N(t)-N(s))}\} = \sum_{k=0}^{\infty} 2^k (t-s)^k/(k!\,\mathrm{e}^{t-s}) = \mathrm{e}^{t-s}$.

Task 6. See page 12 in Stig's lecture notes.