

TMS 165/MSA350 Stochastic Calculus

Solved Exercises for Chapters 2-3 in Klebaner's book

Exercise 1. Prove Equations 2.17 and 2.21 in Klebaner's book for conditional expectations.

Solution. It is an easy exercise to see that any constant random variable (that is, a non-random random variable) is measurable wrt. the trivial σ -field $\{\emptyset, \Omega\}$. In particular, $\mathbf{E}\{X\}$ is $\{\emptyset, \Omega\}$ -measurable. Further we have

$$\int_{\emptyset} \mathbf{E}\{X\} d\mathbf{P} = 0 = \int_{\emptyset} X d\mathbf{P} \quad \text{and} \quad \int_{\Omega} \mathbf{E}\{X\} d\mathbf{P} = \mathbf{P}\{\Omega\} \mathbf{E}\{X\} = \mathbf{E}\{X\} = \int_{\Omega} X d\mathbf{P}.$$

Hence $\mathbf{E}\{X\}$ fulfill the defining properties on page 44 in the book of being the conditional expectation $\mathbf{E}\{X | \{\emptyset, \Omega\}\}$. This establishes (2.17).

As for (2.21), as $\mathbf{E}\{X\}$ is $\{\emptyset, \Omega\}$ -measurable it is measurable wrt. any other σ -field $\mathcal{G} \subseteq \mathcal{F}$ (as any such \mathcal{G} must contain $\{\emptyset, \Omega\}$). For X independent of \mathcal{G} we further have

$$\int_A X d\mathbf{P} = \mathbf{E}\{I_A X\} = \mathbf{E}\{I_A\} \mathbf{E}\{X\} = \mathbf{P}\{A\} \mathbf{E}\{X\} = \int_A \mathbf{E}\{X\} d\mathbf{P} \quad \text{for } A \in \mathcal{G}.$$

Hence $\mathbf{E}\{X\}$ fulfill the defining properties of being the conditional expectation $\mathbf{E}\{X | \mathcal{G}\}$.

Exercise 2. Consider a finite sample space $\Omega = \{1, \dots, 2n\}$ equipped with the σ -field \mathcal{F} consisting of all subsets of Ω together with the uniform probability measure \mathbf{P} on Ω assigning probability $1/(2n)$ to each outcome $\omega \in \Omega$. Calculate $\mathbf{E}\{X | \mathcal{G}\}$ for the random variable $X(\omega) = \omega$ and the σ -field $\mathcal{G} = \{\emptyset, A, A^c, \Omega\}$ where $A = \{1, \dots, n\}$.

Solution. From intuitive reasoning we come up with the hypothesis that

$$\mathbf{E}\{X | \mathcal{G}\} = \begin{cases} (n+1)/2 & \text{for } \omega \in A \\ (3n+1)/2 & \text{for } \omega \in A^c \end{cases}.$$

That this really is correct follows from the fact that this random variable is \mathcal{G} -measurable and that by elementary calculations together with the uniformity of \mathbf{P} it satisfies

$$\int_B \mathbf{E}\{X | \mathcal{G}\} d\mathbf{P} = \int_B X d\mathbf{P} \quad \text{for } B \in \{\emptyset, A, A^c, \Omega\}.$$

Exercise 3. Show that among all zero-mean stochastic processes $\{X(t)\}_{t \geq 0}$ with finite second moments $\mathbf{E}\{X(t)^2\} < \infty$ for $t \geq 0$, the class of martingales contain all processes

with independent increments and are all included among processes with uncorrelated increments.

Solution. For X zero-mean with independent increments we have

$$\mathbf{E}\{X(t)|\mathcal{F}_s^X\} = \mathbf{E}\{X(t)-X(s)|\mathcal{F}_s^X\} + \mathbf{E}\{X(s)|\mathcal{F}_s^X\} = \mathbf{E}\{X(t)-X(s)\} + X(s) = X(s)$$

for $s \leq t$, where we use the independent increments and (2.21) together with the fact that X is adapted to the σ -field $\{\mathcal{F}_t^X\}_{t \geq 0}$. Hence X is a martingale.

On the other hand, for X a zero-mean martingale we have

$$\begin{aligned} \mathbf{E}\{(X(u)-X(t))(X(s)-X(r))\} &= \mathbf{E}\{\mathbf{E}\{(X(u)-X(t))(X(s)-X(r))|\mathcal{F}_s^X\}\} \\ &= \mathbf{E}\{(X(s)-X(r))\mathbf{E}\{X(u)-X(t)|\mathcal{F}_s^X\}\} \\ &= \mathbf{E}\{(X(s)-X(r))(X(s)-X(s))\} \\ &= 0 \end{aligned}$$

for $0 \leq r \leq s \leq t \leq u$, where we made use of Equation 2.20 in Klebaner's book and the fact that X is adapted together with Equation 2.18 and the martingale property.

Exercise 4. Prove Equation 3.4 in Klebaner's book. (Note that it is assumed that $0 < t_1 < \dots < t_n$ in this formula.)

Solution. We prove (3.4) by induction. Note that the property (3.4) when $n = 1$ is just (3.3), which in turn is a rather elementary formula we proved during Lecture 4.

Now assume that (3.4) holds for $n = k$. Note that (3.4) for $n = k$ in turn means that $(B^x(t_1), \dots, B^x(t_k))$ has probability density function

$$f_{(B^x(t_1), \dots, B^x(t_k))}(y_1, \dots, y_k) = p_{t_1}(x, y_1) \prod_{i=2}^k p_{t_i - t_{i-1}}(y_{i-1}, y_i) \quad \text{for } (y_1, \dots, y_k) \in \mathbb{R}^k.$$

For the case when $n = k+1$ it therefore follows from conditioning on the value (y_1, \dots, y_k) of $(B^x(t_1), \dots, B^x(t_k))$ and using independence of increments that

$$\begin{aligned} &\mathbf{P}\left\{\bigcap_{i=1}^{k+1} \{B^x(t_i) \leq x_i\}\right\} \\ &= \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_k} \mathbf{P}\{B^x(t_{k+1}) - B^x(t_k) + y_k \leq x_{k+1}\} f_{(B^x(t_1), \dots, B^x(t_k))}(y_1, \dots, y_k) dy_1 \dots dy_k \\ &= \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_k} \Phi\left(\frac{x_{k+1} - y_k}{\sqrt{t_{k+1} - t_k}}\right) p_{t_1}(x, y_1) \prod_{i=2}^k p_{t_i - t_{i-1}}(y_{i-1}, y_i) dy_1 \dots dy_k \\ &= \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_k} \int_{-\infty}^{x_{k+1}} p_{t_1}(x, y_1) \prod_{i=2}^{k+1} p_{t_i - t_{i-1}}(y_{i-1}, y_i) dy_1 \dots dy_{k+1} \end{aligned}$$

[as $B^x(t_{k+1}) - B^x(t_k)$ is $N(0, t_{k+1} - t_k)$ -distributed]. This proves (3.4) by induction.

Exercise 5. Let ξ and η be independent standard normal random variables. Show that the process $\{X(t)\}_{t \in \{0,1\}}$ given by $X(0) = \text{sign}(\eta) \xi$ and $X(1) = \text{sign}(\xi) \eta$ is not Gaussian despite each of the process values $X(0)$ and $X(1)$ are standard Gaussian.

Solution. It is an elementary exercise to see that $X(0)$ and $X(1)$ are standard Gaussian (normal) distributed. Also note that

$$X(0)X(1) = \text{sign}(\eta) \xi \text{sign}(\xi) \eta = |\xi| |\eta| \geq 0.$$

However, if $(X(0), X(1))$ were bivariate standard Gaussian (as it must be if X is a Gaussian process), then the above non-negativity is possible if and only if $X(0)$ and $X(1)$ have perfect correlation 1. But this is not true as

$$\text{Corr}\{X(0), X(1)\} = \text{Cov}\{X(0), X(1)\} = \mathbf{E}\{X(0)X(1)\} = \mathbf{E}\{|\xi| |\eta|\} = (\mathbf{E}\{|\xi|\})^2 = \frac{2}{\pi} < 1$$

by elementary calculations [where we used that $X(0)$ and $X(1)$ are standard Gaussian].

Exercise 6. Prove that the finite dimensional distributions of a zero-mean Gaussian stochastic process $\{X(t)\}_{t \in T}$ are completely characterized by the covariance function of the process.

Solution. Given $t_1, \dots, t_n \in T$, the distribution of the random variable $(X(t_1), \dots, X(t_n))$ is determined by its characteristic function (Fourier transform)

$$\mathbf{E}\left\{e^{i \sum_{j=1}^n a_j X(t_j)}\right\} \quad \text{for } (a_1, \dots, a_n) \in \mathbb{R}^n.$$

As $\sum_{j=1}^n a_j X(t_j)$ is a univariate zero-mean Gaussian random variable, that characteristic function in turn is equal to

$$\exp\left[-\frac{1}{2} \mathbf{Var}\left\{\sum_{j=1}^n a_j X(t_j)\right\}\right] = \exp\left[-\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n a_i a_j \mathbf{Cov}\{X(t_i), X(t_j)\}\right],$$

which in turn obviously is determined by the covariance function of X .