TMS 165/MSA350 Stochastic Calculus

Solved Exercises for Chapter 4 in Klebaner's book

Througout this exercise session $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Exercise 1. Nobert Wiener defined the stochastic integral process $\{\int_0^t g\,dB\}_{t\geq 0}$ with respect to B for continuously differentiable functions $g:[0,\infty)\to\mathbb{R}$ as

$$\int_0^t g\,dB = g(t)B(t) - \int_0^t B\,dg = g(t)B(t) - \int_0^t B(r)g'(r)\,dr \quad \text{for } t\geq 0.$$

Find the covariance function and mean of $\{\int_0^t g\,dB\}_{t\geq 0}$ defined in this way by means of direct calculation (not using Itô integral theory).

Solution. As mathematical expectation $\mathbf{E}$ commutes with integration $\int$ by Fubini's theorem, we have

$$\mathbf{E}\left\{g(t)B(t) - \int_0^t B(r)g'(r)\,dr\right\} = g(t)\,\mathbf{E}\{B(t)\} - \int_0^t \mathbf{E}\{B(r)\}\,g'(r)\,dr = 0.$$

Assuming that $0\leq s\leq t$ the same reasoning gives

$$\begin{aligned}
&\mathbf{Cov}\left\{g(s)B(s) - \int_0^s B(r_1)g'(r_1)\,dr_1, g(t)B(t) - \int_0^t B(r_2)g'(r_2)\,dr_2\right\}\\
&= \mathbf{E}\left\{\left(g(s)B(s) - \int_0^s B(r_1)g'(r_1)\,dr_1\right)\left(g(t)B(t) - \int_0^t B(r_2)g'(r_2)\,dr_2\right)\right\}\\
&= g(s)g(t)\,\mathbf{E}\{B(s)B(t)\} - g(s)\int_0^t \mathbf{E}\{B(s)B(r)\}\,g'(r)\,dr - g(t)\int_0^s \mathbf{E}\{B(t)B(r)\}\,g'(r)\,dr\\
&\qquad + \int_0^s\int_0^t \mathbf{E}\{B(r_1)B(r_2)\}\,g'(r_1)g'(r_2)\,dr_1dr_2\\
&= g(s)g(t)\,s - g(s)\int_0^s r\,g'(r)\,dr - g(s)\int_s^t s\,g'(r)\,dr - g(t)\int_0^s r\,g'(r)\,dr\\
&\qquad + 2\int_{r_1=0}^{r_1=s}\int_{r_2=r_1}^{r_2=s} r_1\,g'(r_1)g'(r_2)\,dr_1dr_2 + \int_{r_1=0}^{r_1=s}\int_{r_2=s}^{r_2=t} r_1\,g'(r_1)g'(r_2)\,dr_1dr_2\\
&= g(s)^2 s - (g(s)+g(t))\int_0^s r\,g'(r)\,dr\\
&\qquad + 2\,g(s)\int_0^s r\,g'(r)\,dr - 2\int_0^s r\,g'(r)g(r)\,dr + (g(t)-g(s))\int_0^s r\,g'(r)\,dr\\
&= \int_0^s g(r)^2\,dr,
\end{aligned}$$

so that the covariance function is

$$\mathbf{Cov}\left\{\int_0^s g\,dB, \int_0^t g\,dB\right\} = \int_0^{\min\{s,t\}} g(r)^2\,dr \quad \text{for } s,t\geq 0.$$

Exercise 2. Show that convergence in $\mathbb{L}^p$ of random variables for $p \geq 1$ implies convergence in probability as well as that convergence in $\mathbb{L}^p$ of random variables implies convergence of moments of order up to $[p]$.

Solution. If $X_n \to X$ in $\mathbb{L}^p$, then Tjebysjev's inequality shows that

$$\mathbf{P}\big\{|X_n - X| \geq \varepsilon\big\} \leq \frac{\mathbf{E}\big\{|X_n - X|^p\big\}}{\varepsilon^p} \to 0 \quad \text{as } n \to \infty \text{ for } \varepsilon > 0.$$

Further we have by repeated use of Hölder's inequality

$$\begin{aligned}
\big|\mathbf{E}\{X_n^m\} - \mathbf{E}\{X^m\}\big| &= \left|\mathbf{E}\left\{\sum_{i=0}^{m-1} \binom{m}{i} (X_n - X)^{m-i} X^i\right\}\right| \\
&\leq \sum_{i=0}^{m-1} \binom{m}{i} \mathbf{E}\big\{|X_n - X|^{m-i} |X|^i\big\} \\
&\leq \sum_{i=0}^{m-1} \binom{m}{i} \big(\mathbf{E}\big\{|X_n - X|^m\big\}\big)^{(m-i)/m} \mathbf{E}\big\{|X|^m\big\}\big)^{i/m} \\
&\leq \sum_{i=0}^{m-1} \binom{m}{i} \big(\mathbf{E}\big\{|X_n - X|^p\big\}\big)^{(m-i)/p} \mathbf{E}\big\{|X|^p\big\}\big)^{i/p} \\
&\to 0 \quad \text{as } n \to \infty \text{ for } m \leq [p].
\end{aligned}$$

Exercise 3. Show that if $X_n \to X$ in $\mathbb{L}^p$ for an $p \geq 1$ as $n \to \infty$, then $\mathbf{E}\{X_n \,|\, \mathcal{G}\} \to \mathbf{E}\{X \,|\, \mathcal{G}\}$ in $\mathbb{L}^1$ as $n \to \infty$ for any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$.

Solution. This follows using Properties 2.24 and 2.20 of conditional expectation in Klebaner's book together with Hölder's inequality as

$$\begin{aligned}
\mathbf{E}\big\{\big|\mathbf{E}\{X_n \,|\, \mathcal{G}\} - \mathbf{E}\{X \,|\, \mathcal{G}\}\big|\big\} &\leq \mathbf{E}\big\{\mathbf{E}\{|X_n - X| \,|\, \mathcal{G}\}\big\} \\
&= \mathbf{E}\{|X_n - X|\} \\
&\leq \big(\mathbf{E}\{|X_n - X|^p\}\big)^{1/p} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Exercise 4. Show that the Itô integral process $\{\int_0^t X\, dB\}_{t \in [0,T]}$ for $X \in S_T$ is a martingale.

Solution. Pick a partition $0=t_0<t_1<\ldots<t_n=T$ of the interval $[0,T]$ and consider an $X\in S_T$ given by

$$X(t)=I_{\{0\}}(t)\eta_0+\sum_{i=0}^{n-1}I_{(t_i,t_{i+1}]}(t)\xi_i \quad \text{for } t\in[0,T],$$

where η_0 is $\mathcal{F}_0^B$-measurable and ξ_i is $\mathcal{F}_{t_i}^B$-measurable for $i=0,\ldots,n-1$. Recall that

$$\int_0^t X\,dB=\begin{cases}\displaystyle\sum_{i=0}^{m-1}\xi_i\,(B(t_{i+1})-B(t_i))+\xi_m\,(B(t)-B(t_m)) & \text{for } t\in(t_m,t_{m+1}]\\ 0 & \text{for } t=0\end{cases},$$

for $m=0,\ldots,n-1$. As $\mathcal{F}_0^B=\{\emptyset,\Omega\}$ (as $B(0)$ is a constant), we have

$$\mathbf{E}\left\{\int_0^t X\,dB\,\middle|\,\mathcal{F}_0^B\right\}=\mathbf{E}\left\{\int_0^t X\,dB\right\}=0=\int_0^0 X\,dB$$

by Equation 2.17 together with Property 3 of the Itô integral process for S_T in Klebaner's book. Now pick $0<s\le t\le T$ together with integers $0\le k\le m\le n-1$ such that $s\in(t_k,t_{k+1}]$ and $t\in(t_m,t_{m+1}]$. Then we have

$$\begin{aligned}\mathbf{E}\left\{\int_0^t X\,dB\,\middle|\,\mathcal{F}_s^B\right\}&=\mathbf{E}\left\{\sum_{i=0}^{m-1}\xi_i\,(B(t_{i+1})-B(t_i))+\xi_m\,(B(t)-B(t_m))\,\middle|\,\mathcal{F}_s^B\right\}\\&=\sum_{i=0}^{k-1}\xi_i\,(B(t_{i+1})-B(t_i))\\&\quad+\xi_k\,\mathbf{E}\big\{B(t_{k+1})-B(t_k)\big|\mathcal{F}_s^B\big\}\\&\quad+\sum_{i=k+1}^{m-1}\mathbf{E}\big\{\xi_i\,\mathbf{E}\{B(t_{i+1})-B(t_i)\,|\,\mathcal{F}_{t_i}^B\}\big|\mathcal{F}_s^B\big\}\\&\quad+\mathbf{E}\big\{\xi_m\,\mathbf{E}\{B(t)-B(t_m)\,|\,\mathcal{F}_{t_m}^B\}\big|\mathcal{F}_s^B\big\}\\&=\sum_{i=0}^{k-1}\xi_i\,(B(t_{i+1})-B(t_i))+\xi_k\,(B(s)-B(t_k))+0+0\\&=\int_0^s X\,dB \quad \text{for } k<m,\end{aligned}$$

while in the same fashion fashion

$$\begin{aligned}\mathbf{E}\left\{\int_0^t X\,dB\,\middle|\,\mathcal{F}_s^B\right\}&=\sum_{i=0}^{m-1}\xi_i\,(B(t_{i+1})-B(t_i))+\xi_m\,\mathbf{E}\{B(t)-B(t_m)\,|\,\mathcal{F}_s^B\}\\&=\sum_{i=0}^{k-m}\xi_i\,(B(t_{i+1})-B(t_i))+\xi_m\,(B(s)-B(t_m))\\&=\int_0^s X\,dB \quad \text{for } k=m,\end{aligned}$$

Exercise 5. Prove Theorem 4.9 in Klebaner's book for $X \in S_T$.

Solution. Pick a partition $0 = s_0 < s_1 < \ldots < s_m = T$ of the interval $[0, T]$ and consider an $X \in S_T$ given by

$$X(t) = I_{\{0\}}(t)\eta_0 + \sum_{i=0}^{m-1} I_{(s_i, s_{i+1}]}(t)\xi_i \quad \text{for } t \in [0, T],$$

where η_0 is $\mathcal{F}_0^B$-measurable and ξ_i is $\mathcal{F}_{t_i}^B$-measurable for $i = 0, \ldots, m-1$. Now, any given grid $0 = t_0 < t_1 < \ldots < t_n = T$ may be refined to a grid $0 = t'_0 < t'_1 < \ldots < t'_k = T$ with at most $n+m-1$ members that also includes the times $0 < s_1 < \ldots < s_{m-1} < T$. Writing $s_j = t'_{i(j)}$ for $j = 1, \ldots, m-1$, we then have

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\left(\int_0^{t_i} X\,dB - \int_0^{t_{i-1}} X\,dB\right)^2 - \sum_{i=1}^{k}\left(\int_0^{t'_i} X\,dB - \int_0^{t'_{i-1}} X\,dB\right)^2\right| \\
&\quad \le \sum_{j=1}^{m-1}\left(\int_0^{t'_{i(j)+1}} X\,dB - \int_0^{t'_{i(j)-1}} X\,dB\right)^2 \\
&\qquad + \sum_{j=1}^{m-1}\left(\int_0^{t'_{i(j)+1}} X\,dB - \int_0^{t'_{i(j)}} X\,dB\right)^2 \\
&\qquad + \sum_{j=1}^{m-1}\left(\int_0^{t'_{i(j)}} X\,dB - \int_0^{t'_{i(j)-1}} X\,dB\right)^2 \\
&\quad \le 3\sum_{j=1}^{m-1}\left(\int_0^{t'_{i(j)+1}} X\,dB - \int_0^{t'_{i(j)}} X\,dB\right)^2 + 3\sum_{j=1}^{m-1}\left(\int_0^{t'_{i(j)}} X\,dB - \int_0^{t'_{i(j)-1}} X\,dB\right)^2 \\
&\quad \to 0 \quad \text{with probability 1 as } \max_{1\le i\le k} t'_i - t'_{i-1} \le \max_{1\le i\le n} t_i - t_{i-1} \downarrow 0,
\end{aligned}$$

by the continuity of X. To finish the proof it is therefore sufficient to prove that, in the sense of convergence in probability,

$$\lim_{\max_{1\le i\le n} t_i - t_{i-1} \downarrow 0} \sum_{i=1}^{n}\left(\int_0^{t_i} X\,dB - \int_0^{t_{i-1}} X\,dB\right)^2 = \int_0^T X(r)^2\,dr$$

for grids $0 = t_0 < t_1 < \ldots < t_n = T$ that include the times $0 < s_1 < \ldots < s_{m-1} < T$. That this is so in turn is an immediate consequence of the fact that $[B]([s,t]) = t - s$.

Exercise 6. Show that the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ for $X \in E_T$ is a martingale.

Solution. Picking $\{X_n\}_{n=1}^{\infty} \subseteq S_T$ such that

$$\mathbf{E}\bigg\{\int_0^T (X_n(t)-X(t))^2\,dt\bigg\} \to 0 \quad \text{as } n\to\infty,$$

the definition of the Itô integral process for E_T together with Exercise 3 and the martingale property for the Itô integral process for S_T (recall Exercise 4) show that

$$\mathbf{E}\bigg\{\int_0^t X\,dB \,\bigg|\, \mathcal{F}_s^B\bigg\} \leftarrow \mathbf{E}\bigg\{\int_0^t X_n\,dB \,\bigg|\, \mathcal{F}_s^B\bigg\} = \int_0^s X_n\,dB \to \int_0^s X\,dB \quad \text{for } 0\le s\le t\le T$$

in the sense of convergence in $\mathbb{L}^1$ (for the first limit) and in the sense of convergence in $\mathbb{L}^2$ (for the second limit), respectively, as $n\to\infty$. Now the conditional expectation on the left hand side in the above equation must be equal to something, which in turn can therefore only be the integral on the right hand side.