TMS165/MSA350 Stochastic Calculus

Written Exam Tuesday 28 October 2014 8.30–12.30 am

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GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Show the Stratanovich chain rule $\partial(X(t)Y(t)) = X(t)\partial Y(t) + Y(t)\partial X(t)$ for Itô processes $X(t)$ and $Y(t)$. (Here ∂ is the Stratanovich differential.) **(5 points)**

Task 2. Let $X(t)$ and $Y(t)$ be strictly positive Itô processes with $X(0) = Y(0) = 1$. Is it true that $\mathcal{L}(XY)(t) = \mathcal{L}(X)(t) + \mathcal{L}(Y)(t)$? (Here $\mathcal{L}$ is the stochastic logarithm.) **(5 points)**

Task 3. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables with mean function $\mu_n = \mathbf{E}\{X_n\}$ and covariance function $r_{m,n} = \mathbf{Cov}\{X_m, X_n\}$ such that $r_{n,n} = 1$ for all n. Under what conditions on μ_n and $r_{m,n}$ do we have $\mathbf{E}\{(X_n - X)^2\} \to 0$ as $n \to \infty$ for some random variable X with $\mathbf{E}\{X^2\} < \infty$? **(5 points)**

Task 4. Let $f(x,t)$ be a function that has continuous first and second order partial derivatives. Under what additional conditions on $f(x,t)$ is $\{B(t)^2 + f(B(t),t)\}_{t\in[0,T]}$ a martingale? **(5 points)**

Task 5. Find a PDE that is solved by the fair price $p(x,t) = \mathbf{E}\big\{\max\{X(T)-K,0\}\big| X(t) = x\}$ of an European call option at time $t \in [0,T)$ for an interest rate process $\{X(t)\}_{t\in[0,T]}$ given by the Vasicek SDE $dX(t) = a\,(b - X(t))\,dt + c\,dB(t)$ for $t \in [0,T]$, $X(0) = x_0$. (Here $K, a, c > 0$ and $b, x_0 \in \mathbb{R}$ are givenm constants.) **(5 points)**

Task 6. Given sufficiently nice functions $\mu(x)$ and $\sigma(x)$ together with a constant $\lambda \in (0,1)$ a composite Euler scheme for calculation of a numerical approximation Y_N of the exact value $X(T)$ of the solution of the SDE $dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t)$ for $t \in (0,T]$, $X(0) = X_0$, at $t = T$ is given by $Y_0 = X_0$ together with the recursive scheme

$$Y_n = Y_{n-1} + \big(\mu(Y_n) - f(Y_n)\big)(t_n - t_{n-1}) + \big(\lambda\sigma(Y_n) + (1-\lambda)\sigma(Y_{n-1})\big)(B(t_n) - B(t_{n-1}))$$

for $n = 1,\ldots,N$, where $0 = t_0 < t_1 < \ldots < t_N = T$. Find the function $f(x)$ that makes Y_N converge to $X(T)$ as $\max_{1\leq n\leq N} t_n - t_{n-1} \downarrow 0$. **(5 points)**

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Solutions to Written Exam 28 October 2014

Task 1. This is Theorem 5.18 in Klebaner's book.

Task 2. By Theorem 5.3 in Klebaner's book we see that $\mathcal{L}(XY)(t) = \int_0^t \frac{d[XY,XY](t)}{2\,X(t)^2Y(t)^2} + \ln(X(t)Y(t)) = \int_0^t \frac{(d(X(t)Y(t)))^2}{2\,X(t)^2Y(t)^2} + \ln(X(t)) + \ln(Y(t)) = \int_0^t \frac{(X(t)dY(t)+Y(t)dX(t)+d[X,Y](t))^2}{2\,X(t)^2Y(t)^2} + \ln(X(t)) + \ln(Y(t)) = \int_0^t \frac{(X(t)dY(t)+Y(t)dX(t))^2}{2\,X(t)^2Y(t)^2} + \ln(X(t)) + \ln(Y(t)) = \int_0^t \frac{dX(t)^2}{2\,X(t)^2} + \ln(X(t)) + \int_0^t \frac{dY(t)^2}{2\,Y(t)^2} + \ln(Y(t)) + \int_0^t \frac{dX(t)dY(t)}{X(t)Y(t)} = \mathcal{L}(X)(t) + \mathcal{L}(Y)(t) + \int_0^t \frac{d[X,Y](t)}{X(t)Y(t)}$, which is not equal to $\mathcal{L}(X)(t) + \mathcal{L}(Y)(t)$ in general.

Task 3. According to the Cauchy criterion we have the requested convergence if and only if $\mathbf{E}\{(X_m - X_n)^2\} = \mathbf{Var}\{X_m - X_n\} + (\mathbf{E}\{X_m - X_n\})^2 = 2\,(1 - r_{m,n}) + (\mu_m - \mu_n)^2 \to 0$ as $m,n \to \infty$, which is to say that $r_{m,n} \to 1$ as $m,n \to \infty$ and $\mu_n \to \mu$ for some $\mu \in \mathbb{R}$ as $n \to \infty$ (making use also of the Cauchy criterion for real sequences).

Task 4. As $d(B(t)^2) = 2\,B(t)\,dB(t) + dt$, where $2\int_0^t B(s)\,dB(s)$ is a martingale, we see that $B(t)^2 + f(B(t),t)$ is a martingale when $g(B(t),t) = f(B(t),t) + t$ is. By Itô's formula that in turn holds when $\frac{1}{2}\frac{\partial^2}{\partial x^2} g(x,t) + \frac{\partial}{\partial t} g(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(x,t) + \frac{\partial}{\partial t} f(x,t) + 1 = 0$ and $\int_0^T (\frac{\partial}{\partial x} f(B(t),t))^2\,dt < \infty$, because then we have $B(t)^2 + f(B(t),t) = \int_0^t \big(2\,B(s) + \frac{\partial}{\partial x} f(B(s),s)\big)\,dB(s)$ with $\{2\,B(t) + \frac{\partial}{\partial x} f(B(t),t)\}_{t\in[0,T]} \in E_T$.

Task 5. From Theorem 6.7 in Klebaner's book it follows that the PDE is

$$L_t p(x,t) + \frac{\partial p(x,t)}{\partial t} = \frac{c^2}{2}\,\frac{\partial^2 p(x,t)}{\partial x^2} + a\,(b - x)\,\frac{\partial p(x,t)}{\partial x} + \frac{\partial p(x,t)}{\partial t} = 0 \quad \text{for } t \in [0,T)$$

with $p(x,T) = \max\{x - K, 0\}$.

Task 6. According to the Euler method we have the convergence

$$\sum_{n=1}^{N} \big(\mu(Y_{n-1})\,(t_n - t_{n-1}) + \sigma(Y_{n-1})\,(B(t_n) - B(t_{n-1}))\big) \to X(T)$$

as $\max_{1\leq n\leq N} t_n - t_{n-1} \downarrow 0$. Here we have

$$\mu(Y_{n-1})\,(t_n - t_{n-1}) \approx \big(\mu(Y_n) - \mu'(Y_n)\,(Y_n - Y_{n-1})\big)\,(t_n - t_{n-1}) \approx \mu(Y_n)\,(t_n - t_{n-1})$$

and

$$\begin{aligned}\sigma(Y_{n-1})\,(B(t_n) - B(t_{n-1})) &\approx \big(\sigma(Y_n) - \sigma'(Y_n)\,(Y_n - Y_{n-1})\big)\,(B(t_n) - B(t_{n-1})) \\ &\approx \sigma(Y_n)\,(B(t_n) - B(t_{n-1})) - \sigma'(Y_n)\sigma(Y_n)\,(B(t_n) - B(t_{n-1}))^2 \\ &\approx \sigma(Y_n)\,(B(t_n) - B(t_{n-1})) - \sigma'(Y_n)\sigma(Y_n)\,(t_n - t_{n-1}),\end{aligned}$$

so that we must have $f(x) = \lambda\,\sigma'(x)\sigma(x)$.