TMS165/MSA350 Stochastic Calculus

Written Exam Wednesday 15 April 2015 8.30 – 12.30 am

TEACHER AND JOUR: Patrik Albin, telephone 070 6945709. AIDS: None.

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Prove that $[B,B]([0,t]) = t$. **(5 points)**

Task 2. Prove the isometry property $\mathbf{E}\big\{(\int_0^T X(t)\,dB(t))^2\big\} = \int_0^T \mathbf{E}\{X(t)^2\}\,dt$ for the Itô integral $\int_0^T X(t)\,dB(t)$ of a simple adapted process $\{X(t)\}_{t\in[0,T]}$. **(5 points)**

Task 3. Solve the following (non-diffusion type) SDE

$$dX(t) = B(t)\,X(t)\,dt + B(t)\,X(t)\,dB(t) \quad \text{for } t\geq 0, \quad X(0)=1. \qquad \textbf{(5 points)}$$

Task 4. Show that if $\{X(t)\}_{t\in[0,T]}$ is a solution to the SDE

$$\mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t) \quad \text{for } t\geq 0, \quad X(0)=x_0,$$

and if $f(x,t)$ is a solution to the PDE

$$\frac{\sigma(x,t)^2}{2}\,\frac{\partial^2 f(x,t)}{\partial x^2} + \mu(x,t)\,\frac{\partial f(x,t)}{\partial x} + \frac{\partial f(x,t)}{\partial t} = 0 \quad \text{for } t\in[0,T], \quad f(x,T)=g(x),$$

then we have (under additional technical conditions) $f(x,t) = \mathbf{E}\{g(X(T))\,|\,X(t)=x\}$.

(5 points)

Task 5. Let $\mathcal{G}$ be a sub-σ-field of a σ-field $\mathcal{F}$ on which two probability measures $\mathbf{Q}$ and $\mathbf{P}$ are defined. Show that if $\mathbf{Q}$ is absolutely continuous with respect to $\mathbf{P}$ with $d\mathbf{Q} = \Lambda\,d\mathbf{P}$ and if X is a $\mathbf{Q}$-integrable random variable, then ΛX is $\mathbf{P}$-integrable and $\mathbf{E}_{\mathbf{Q}}\{X|\mathcal{G}\} = \mathbf{E}_{\mathbf{P}}\{\Lambda X|\mathcal{G}\}/\mathbf{E}_{\mathbf{P}}\{\Lambda|\mathcal{G}\}$. **(5 points)**

Task 6. Explain the difference between strong convergence and weak convergence of a numerical method for solution of SDE's. **(5 points)**

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Solutions to Written Exam 15 April 2015

Task 1. See the proof of Theorem 3.4 in Klebaner's book.

Task 2. The proof is done on page 94 in Klebaner's book.

Task 3. This is Exercise 5.3 in Klebaner's book – according to his solution we have $X(t) = \exp\big\{ \int_0^t (B(s) - \frac{1}{2}B(s)^2)\, ds + \int_0^t B(s)\, dB(s) \big\}$.

Task 4. See the proof of Theorem 6.6 in Klebaner's book.

Task 5. See the proof of Theorem 10.8 in Klebaner's book.

Task 6. See the beginning of Chapter 2 in the lecture notes by Stig Larsson.