TMS 165/MSA350 Stochastic Calculus

Solved Exercises for Chapters 5-6 and 10 in Klebaner's book

Througout this exercise session $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Exercise 1. Show that

$$X(t) = \mathrm{e}^{-\alpha t}\bigg(\frac{\sigma}{\sqrt{2\alpha}}\big(B(\mathrm{e}^{2\alpha t}) - B(1)\big) + x_0\bigg) \quad \text{for } t \geq 0$$

is an Ornstein-Uhlenbeck process in the sense that it got the same distributional properties (finite dimensional distributions) as the solution

$$\{X(t)\}_{t\geq 0} = \bigg\{\mathrm{e}^{-\alpha t}\bigg(x_0 + \sigma\int_0^t \mathrm{e}^{\alpha r}\, dB(r)\bigg)\bigg\}_{t\geq 0}$$

to the Langevin SDE

$$dX(t) = -\alpha\, X(t)\, dt + \sigma\, dB(t) \quad \text{for } t > 0, \quad X(0) = x_0,$$

where $\alpha, \sigma > 0$ and $x_0 \in \mathbb{R}$ are constants.

Solution. As both the above X processes are Gaussian they have the same finite dimensional distributions if their mean and covariance functions agree. Here we clearly have $\mathbf{E}\{X(t)\} = \mathrm{e}^{-\alpha t}x_0$ for $t \geq 0$ for both the X processes. Further, we have

$$\begin{aligned}
\mathbf{Cov}\{X(s), X(t)\} &= \frac{\sigma^2}{2\alpha}\,\mathrm{e}^{-\alpha(s+t)}\,\mathbf{Cov}\big\{B(\mathrm{e}^{2\alpha s}) - B(1), B(\mathrm{e}^{2\alpha t}) - B(1)\big\} \\
&= \frac{\sigma^2}{2\alpha}\,\mathrm{e}^{-\alpha(s+t)}\,\big(\mathrm{e}^{2\alpha\min\{s,t\}} - 1 - 1 + 1\big) \\
&= \frac{\sigma^2}{2\alpha}\,\big(\mathrm{e}^{-\alpha|s-t|} - \mathrm{e}^{-\alpha(s+t)}\big) \quad \text{for } s, t \geq 0
\end{aligned}$$

for the first X process, while Theorem 4.11 in Klebaner's book shows that

$$\mathbf{Cov}\{X(s), X(t)\} = \sigma^2\,\mathrm{e}^{-\alpha(s+t)}\int_0^{\min\{s,t\}} \mathrm{e}^{2\alpha r}\, dr = \frac{\sigma^2}{2\alpha}\,\big(\mathrm{e}^{-\alpha|s-t|} - \mathrm{e}^{-\alpha(s+t)}\big)$$

for $s, t \geq 0$ for the second X process.

Exercise 2. Use the expression for an Ornstein Uhlenbeck process expressed in terms of B from Exercise 1 to find the transition density function for the solution to the Langevin SDE (the Ornstein Uhlenbeck process).

Solution. We have

$$\begin{aligned}
X(t+s) &= \mathrm{e}^{-\alpha(t+s)}\bigg(\frac{\sigma}{\sqrt{2\alpha}}\big(B(\mathrm{e}^{2\alpha(t+s)})-B(1)\big)+x_0\bigg)\\
&= \mathrm{e}^{-\alpha(t+s)}x_0+\frac{\sigma}{\sqrt{2\alpha}}\,\mathrm{e}^{-\alpha(t+s)}\Big(\big(B(\mathrm{e}^{2\alpha(t+s)})-B(\mathrm{e}^{2\alpha s})\big)+\big(B(\mathrm{e}^{2\alpha s})-B(1)\big)\Big)\\
&= \frac{\sigma}{\sqrt{2\alpha}}\,\mathrm{e}^{-\alpha(t+s)}\big(B(\mathrm{e}^{2\alpha(t+s)})-B(\mathrm{e}^{2\alpha s})\big)+\mathrm{e}^{-\alpha t}X(s),
\end{aligned}$$

where

$$\frac{\sigma}{\sqrt{2\alpha}}\,\mathrm{e}^{-\alpha(t+s)}\big(B(\mathrm{e}^{2\alpha(t+s)})-B(\mathrm{e}^{2\alpha s})\big)$$

is an $\mathrm{N}\big(0,(\sigma^2/(2\alpha))\,(1-\mathrm{e}^{-2\alpha t})\big)$-distributed random variable independent of $\{X(r)\}_{r\le s}$. It follows that $(X(t+s)\,|X(s)=x)$ is $\mathrm{N}\big(\mathrm{e}^{-\alpha t}x,(\sigma^2/(2\alpha))\,(1-\mathrm{e}^{-2\alpha t})\big)$-distributed, so that

$$p(y,t+s,x,s)=\frac{d}{dy}\,P(y,t+s,x,s)=\frac{\sqrt{\alpha}}{\sqrt{\pi\,(1-\mathrm{e}^{-2\alpha t})}\,\sigma}\,\exp\bigg\{-\frac{\alpha\,(y-x\,\mathrm{e}^{-\alpha t})^2}{\sigma^2\,(1-\mathrm{e}^{-2\alpha t})}\bigg\}$$

for $t+s>s\ge 0$ and $x,y\in\mathbb{R}$.

Exercise 3. Solve the Stratanovich SDE

$$dX(t)=-\alpha\,dt+\sigma\,X(t)\,\partial B(t)\quad\text{for } t>0,\quad X(0)=x_0,$$

where $\alpha,\sigma>0$ and $x_0\in\mathbb{R}$ are constants.

Solution. By Theorem 5.20 in Klebaner's book the above SDE is equivalent to the Itô SDE

$$dX(t)=\big(\tfrac{1}{2}\,\sigma^2\,X(t)-\alpha\big)\,dt+\sigma\,X(t)\,dB(t)\quad\text{for } t>0,\quad X(0)=x_0.$$

This in turn is a rather simple form of the linear SDE treated in Section 5.3 in Klebaner's book, with a solution given by

$$X(t)=U(t)\bigg(x_0-\alpha\int_0^t\frac{ds}{U(s)}\bigg)\quad\text{where}\quad U(t)=\mathrm{e}^{\sigma B(t)},$$

which is to say that

$$X(t)=x_0\,\mathrm{e}^{\sigma B(t)}-\alpha\,\mathrm{e}^{\sigma B(t)}\int_0^t\mathrm{e}^{-\sigma B(s)}\,ds\quad\text{for } t\ge 0.$$

Exercise 4. The CKLS (Chan-Koralyi-Longstaff-Sanders) SDE is given by

$$dX(t)=(\alpha+\beta\,X(t))\,dt+\sigma\,X(t)^\gamma\,dB(t)\quad\text{for } t>0,\quad X(0)=x_0,$$

where $\alpha, \sigma, \gamma, x_0 > 0$ and $\beta \in \mathbb{R}$ are constants. This SDE is used in contemporary mathematical finance research as a model for, e.g., interest rates and/or deseasonalized eletricity prices, and is famous for being very hard to do inference for and very hard to simulate when $\gamma > 1$. Determine the stationary distribution for this SDE when it exists.

Solution. First note that the fact that $\alpha, x_0 > 0$ ensures that the solution is strictly positive when it exists. From Equation 6.69 in Klebaner's book we further see that the stationary probability density function is given by

$$\pi(x) = \frac{1}{C\, x^{2\gamma}} \exp\biggl\{\int_1^x \frac{2\,(\alpha+\beta\, y)}{\sigma^2\, y^{2\gamma}}\, dy\biggr\} \quad \text{for } x > 0,$$

whenever this function can be normalized to become a density, that is, whenever

$$C = \int_0^\infty \frac{1}{x^{2\gamma}} \exp\biggl\{\int_1^x \frac{2\,(\alpha+\beta\, y)}{\sigma^2\, y^{2\gamma}}\, dy\biggr\}\, dx < \infty.$$

The issue whether C is finite or not in turn clearly boils down to check the integrability properties of the function

$$f(x) = \frac{1}{x^{2\gamma}} \exp\biggl\{\int_1^x \frac{2\,(\alpha+\beta\, y)}{\sigma^2\, y^{2\gamma}}\, dy\biggr\}$$

as $x \downarrow 0$ and as $x \uparrow \infty$. Now, as $x \downarrow 0$ we see that

$$f(x) \sim \begin{cases} C_1\, x^{-2\gamma} & \text{for } \gamma \in (0, 1/2), \\ C_2\, x^{2\alpha/\sigma^2 - 1} & \text{for } \gamma = 1/2, \\ C_3\, x^{-2\gamma} \exp\{-(2\alpha/(\sigma^2(2\gamma-1)))\, x^{-(2\gamma-1)}\} & \text{for } \gamma > 1/2, \end{cases}$$

where $C_1, C_2, C_3 > 0$ are constants. This is to say that we always have the integrability required as $x \downarrow 0$. When $x \uparrow \infty$ we further see that

$$f(x) \sim \begin{cases} C_4\, x^{-2\gamma} & \text{for } \gamma > 1, \\ C_5\, x^{-2+2\beta/\sigma^2} & \text{for } \gamma = 1, \\ C_6\, x^{-2\gamma} \exp\{(\beta/(\sigma^2(1-\gamma)))\, x^{2-2\gamma}\} & \text{for } \gamma \in (1/2, 1), \\ C_7\, x^{2\alpha/\sigma^2 - 1} \exp\{(2\beta/\sigma^2)\, x\} & \text{for } \gamma = 1/2, \\ C_8\, x^{-2\gamma} \exp\{(\beta/(\sigma^2(1-\gamma)))\, x^{2-2\gamma} + (2\alpha/(\sigma^2(1-2\gamma)))\, x^{1-2\gamma}\} & \text{for } \gamma \in (0, 1/2), \end{cases}$$

where $C_4, \ldots, C_8 > 0$ are constants. This is to say that we have the integrability required when

$$\gamma > 1 \quad \text{and} \quad \gamma = 1,\ 2\beta < \sigma^2 \quad \text{and} \quad \gamma \in (1/2, 1),\ \beta \leq 0 \quad \text{and} \quad \gamma \in (0, 1/2],\ \beta < 0.$$

Exercise 5. Exercise 6.10 in Klebaner's book.

Solution. See the solution at the end of Klebaner's book.

Exercise 6. Let X be a standard normal distributed random variable. Show how X can be made to have any given probability density function $f:\mathbb{R}\to[0,\infty)$ by means of a change of probability measure. Also, if X has probability density function $f:\mathbb{R}\to[0,\infty)$, is it possible to make X have standard normal distributed by a change of probability measure?

Solution. Clearly X has probability density function f under the probability measure

$$\mathbf{Q}(A)=\int_A f(X)\,\sqrt{2\pi}\,\mathrm{e}^{X^2/2}\,d\mathbf{P}\quad\text{for }\ A\in\mathcal{F},$$

as this gives

$$\begin{aligned}\mathbf{Q}\{X\in B\}&=\mathbf{E_Q}\big\{I_{\{X\in B\}}\big\}\\&=\mathbf{E_P}\big\{I_{\{X\in B\}}\,f(X)\,\sqrt{2\pi}\,\mathrm{e}^{X^2/2}\big\}\\&=\int_{\mathbb{R}}I_B(x)\,f(x)\,\sqrt{2\pi}\,\mathrm{e}^{x^2/2}\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}\,dx\\&=\int_B f(x)\,dx\quad\text{for }\ B\subseteq\mathbb{R}.\end{aligned}$$

If X instead has a strictly positive probability density function $f:\mathbb{R}\to(0,\infty)$ from the beginning, then X is standard normal distributed under the probability measure

$$\mathbf{Q}(A)=\int_A\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-X^2/2}\,\frac{1}{f(X)}\,d\mathbf{P}\quad\text{for }\ A\in\mathcal{F},$$

as this gives

$$\begin{aligned}\mathbf{Q}\{X\in B\}&=\mathbf{E_Q}\big\{I_{\{X\in B\}}\big\}\\&=\mathbf{E_P}\left\{I_{\{X\in B\}}\,\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-X^2/2}\,\frac{1}{f(X)}\right\}\\&=\int_{\mathbb{R}}I_B(x)\,\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}\,\frac{1}{f(x)}\,f(x)\,dx\\&=\int_B\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}\,dx\quad\text{for }\ B\subseteq\mathbb{R}.\end{aligned}$$

If f is not strictly positive, then it is not possible to make X standard normal distributed by means of this approach, as we then have

$$
\begin{aligned}
\mathbf{Q}\{\Omega\} &= \mathbf{E_Q}\{1\} \\
&= \mathbf{E_P}\left\{ \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-X^2/2}\, \frac{1}{f(X)} \right\} \\
&= \int_{\{x\in\mathbb{R}:f(x)>0\}} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2}\, \frac{1}{f(x)}\, f(x)\, dx \\
&= \int_{\{x\in\mathbb{R}:f(x)>0\}} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2}\, dx \\
&< 1,
\end{aligned}
$$

so that **Q** is no longer a probability measure.