TMS 165/MSA350 Stochastic Calculus

Solved Exercises on Numerical Methods

Exercise 1. Exercise 1 in Stig Larsson's lecture notes "Numerical Methods for Stochastic ODEs".

Solution. The task is to prove the estimate (8) given the conditions (1)-(5) and with Grönwall's lemma and Doob's inequality at our disposal. To land the proof we will assume that $X \in E_T$, so that the Itô integral involved is a square-integrable martingale: Without this additional assumption (which can be waived at the cost of involving the much deeper Burkholder-Davis-Gundy inequality for local martingales in Section 7.7 of Klebaner's book) we neither know that the Itô integral involved is a martingale or that the isometry formula can be used, which is not what is really meant:) Now the proof is just a version of that of Theorem 3, as we have

$$
\begin{aligned}
&\frac{1}{3}\,\mathbf{E}\Big\{\sup_{s\in[0,t]} X(s)^2\Big\}\\
&= \frac{1}{3}\,\mathbf{E}\bigg\{\sup_{s\in[0,t]}\Big(X_0+\int_0^s \mu(X(r),r)\,dr+\int_0^s \sigma(X(r),r)\,dB(r)\Big)^2\bigg\}\\
&\le \mathbf{E}\{X_0^2\}+\mathbf{E}\bigg\{\sup_{s\in[0,t]}\Big(\int_0^s L\,(1+|X(r)|)\,dr\Big)^2\bigg\}+\mathbf{E}\bigg\{\sup_{s\in[0,t]}\Big(\int_0^s \sigma(X(r),r)\,dB(r)\Big)^2\bigg\}\\
&\le \mathbf{E}\{X_0^2\}+\mathbf{E}\bigg\{\int_0^t L^2\,(1+|X(r)|)^2\,dr\int_0^t 1^2\,dr\bigg\}+\Big(\frac{2}{2-1}\Big)^2\,\mathbf{E}\bigg\{\Big(\int_0^t \sigma(X(r),r)\,dB(r)\Big)^2\bigg\}\\
&\le \mathbf{E}\{X_0^2\}+2\,L^2\,T^2+2\,L^2\,T\int_0^t \mathbf{E}\{X(r)^2\}\,dr+4\int_0^t \mathbf{E}\{\sigma(X(r),r)^2\}\,dr\\
&\le \mathbf{E}\{X_0^2\}+2\,L^2\,T^2+2\,L^2\,T\int_0^t \mathbf{E}\{X(r)^2\}\,dr+8\,L^2\,T+8\,L^2\int_0^t \mathbf{E}\{X(r)^2\}\,dr\\
&\le \mathbf{E}\{X_0^2\}+2\,L^2\,T^2+8\,L^2\,T+(2\,L^2\,T+8\,L^2)\int_0^t \sup_{r\in[0,s]}\mathbf{E}\{X(r)^2\}\,dr\\
&\le \Big(\mathbf{E}\{X_0^2\}+2\,L^2\,T^2+8\,L^2\,T\Big)\,\exp\Big\{\big(2\,L^2\,T+8\,L^2\big)\,t\Big\}\\
&\le \Big(\mathbf{E}\{X_0^2\}+2\,L^2\,T^2+8\,L^2\,T\Big)\,\exp\Big\{\big(2\,L^2\,T+8\,L^2\big)\,T\Big\}\quad \text{for } t\in[0,T].
\end{aligned}
$$

Exercise 2. Exercise 2 in Larsson's lecture notes.

Solution. Consider the operator $E_T \ni Y \to G(Y) \in E_T$ given by

$$
G(Y)(t) = X_0+\int_0^t \mu(Y(r),r)\,dr+\int_0^t \sigma(Y(r),r)\,dB(r)\quad \text{for } t\in[0,T].
$$

Note that Exercise 1 ensures that G maps E_T on E_T. Further, note that E_T is a Banach space when equipped with the norm $\|X\| = \sup_{t\in[0,T]} \sqrt{E\{X(t)^2\}}$. By a version of the argument employed in the solution of Exercise 1 we have

$$
\begin{aligned}
&\frac{1}{2}\,\|G(Y)-G(Z)\|^2\\
&\leq \frac{1}{2}\,\mathbf{E}\Big\{\sup_{t\in[0,T]}(G(Y)(t)-G(Z)(t))^2\Big\}\\
&= \frac{1}{2}\,\mathbf{E}\bigg\{\sup_{t\in[0,T]}\bigg(\int_0^t(\mu(Y(s),s)-\mu(Z(s),s))\,ds+\int_0^t(\sigma(Y(s),s)-\sigma(Z(s),s))\,dB(s)\bigg)^2\bigg\}\\
&\leq \mathbf{E}\bigg\{\sup_{t\in[0,T]}\bigg(\int_0^t L\,|Y(s)-Z(s)|\,ds\bigg)^2\bigg\}+\mathbf{E}\bigg\{\sup_{t\in[0,T]}\bigg(\int_0^t(\sigma(Y(s),s)-\sigma(Z(s),s))\,dB(s)\bigg)^2\bigg\}\\
&\leq \mathbf{E}\bigg\{\int_0^T L^2\,|Y(s)-Z(s)|^2\,ds\int_0^T 1^2\,ds\bigg\}+4\,\mathbf{E}\bigg\{\bigg(\int_0^T(\sigma(Y(s),s)-\sigma(Z(s),s))\,dB(s)\bigg)^2\bigg\}\\
&\leq L^2\,T\int_0^T\mathbf{E}\big\{|Y(s)-Z(s)|^2\big\}\,ds+4\int_0^T\mathbf{E}\big\{(\sigma(Y(s),s)-\sigma(Z(s),s))^2\big\}\,ds\\
&\leq (L^2\,T+4L^2)\int_0^T\mathbf{E}\big\{|Y(s)-Z(s)|^2\big\}\,ds\\
&\leq (L^2\,T^2+4L^2T)\,\|Y-Z\|^2.
\end{aligned}
$$

Hence G is a contraction so that the fixed point theorem for contarctions on Banach spaces ensures that there exists a unique $X\in E_T$ such that $G(X)=X$. This in turn is of course a unique strong solution to the SDE (1).

Exercise 3. Exercise 3 in Larsson's lecture notes.

Solution. By a version of the argument employed in the solution of Exercise 1 we have

$$
\begin{aligned}
&\frac{1}{3}\,\mathbf{E}\big\{(\hat{X}(t)-X(t))^2\big\}\\
&= \frac{1}{3}\,\mathbf{E}\bigg\{\bigg(\hat{X}_0-X_0+\int_0^t(\mu(\hat{X}(s),s)-\mu(X(s),s))\,ds+\int_0^t(\sigma(\hat{X}(s),s)-\sigma(X(s),s))\,dB(s)\bigg)^2\bigg\}\\
&\leq \mathbf{E}\big\{(\hat{X}_0-X_0)^2\big\}+\mathbf{E}\bigg\{\bigg(\int_0^t L\,|\hat{X}(s)-X(s)|\,ds\bigg)^2\bigg\}+\mathbf{E}\bigg\{\bigg(\int_0^t(\sigma(\hat{X}(s),s)-\sigma(X(s),s))\,dB(s)\bigg)^2\bigg\}\\
&\leq \mathbf{E}\big\{(\hat{X}_0-X_0)^2\big\}+L^2T\int_0^t\mathbf{E}\big\{(\hat{X}(s)-X(s))^2\big\}\,ds+L^2\int_0^t\mathbf{E}\big\{(\hat{X}(s)-X(s))^2\big\}\,ds\\
&\leq \mathbf{E}\big\{(\hat{X}_0-X_0)^2\big\}\,\exp\big\{L^2(1+T)\,t\big\}\\
&\leq \mathbf{E}\big\{(\hat{X}_0-X_0)^2\big\}\,\exp\big\{L^2(1+T)\,T\big\}\quad\text{for }\ t\in[0,T].
\end{aligned}
$$

Exercise 4. Given some constants $\mu,\sigma\in\mathbb{R}$, consider the SDE

$$dX(t)=\mu\,dt+\sigma\,X(t)\,dB(t)\quad\text{for }\ t\in(0,T],\qquad X(0)=X_0.$$

(a) Show that the unique strong solution to this SDE is given by

$$X(t) = \mathrm{e}^{\sigma B(t) - \sigma^2 t/2}\left(X_0 + \mu\int_0^t \mathrm{e}^{-\sigma B(s) + \sigma^2 s/2}\, ds\right) \quad \text{for } \ t \in [0, T].$$

(b) Solve the SDE numerically for $\mu = \sigma = 1$, $T = 10$ and $X_0 = 0$ by the Euler method. Plot a sample path of the numerical solution and compare with the analytic solution.

Solution. **(a)** According to Theorem 5.4 in Klebaner's book the SDE has a unique strong solution. By application of Itô's formula Theorem 4.17 in Klebaner's book with

$$f(y,z) = y\,z, \quad Y(t) = \mathrm{e}^{\sigma B(t) - \sigma^2 t/2} \quad \text{and} \quad Z(t) = X_0 + \mu\int_0^t \mathrm{e}^{-\sigma B(t) + \sigma^2 s/2}\, ds$$

(cf. Example 4.25 in Klebaner's book) we further see that $Y(t)Z(t)$ is the solution as

$$\begin{aligned} d(Y(t)Z(t)) &= Y(t)\,dZ(t) + Z(t)\,dY(t) + dY(t)dZ(t) \\ &= \mu\,dt + Y(t)Z(t)\,\sigma\,dB(t) + (\mu\,dt)\,(\sigma\,dB(t)) \\ &= \mu\,dt + Y(t)Z(t)\,\sigma\,dB(t). \end{aligned}$$

(b)

```
In[25]:= Clear[steps, mu, sigma, T, X0, dB, Xnumeric, B, IntB, Xanalytic]; steps = 10000; mu = 1; sigma = 1;
         T = 10; X0 = 0; dB = Table[Random[NormalDistribution[0, Sqrt[T/steps]]], {i, 1, steps}];

In[35]:= For[i = 1; Xnumeric = {X0}, i ≤ steps, i++,
          AppendTo[Xnumeric, Xnumeric[[i]] + mu * T/steps + sigma * Xnumeric[[i]] * dB[[i]]]];
         ListPlot[Xnumeric, PlotJoined → True, PlotRange → {-0.51, 20.1},
          Ticks → {{{1000, ""}, {2000, "2"}, {3000, ""}, {4000, "4"}, {5000, ""},
             {6000, "6"}, {7000, ""}, {8000, "8"}, {9000, ""}, {10000, "10"}}, Automatic}]
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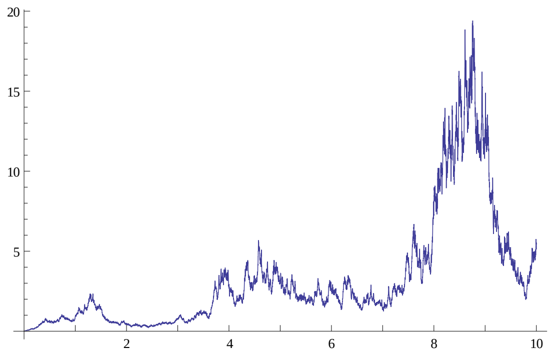
Out[35]=

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In[36]:= For[i = 1; Xanalytic = {X0}; B = {0}; IntB = {0}, i ≤ steps, i++, AppendTo[B, B[[i]] + dB[[i]]];
          AppendTo[IntB, IntB[[i]] + Exp[-sigma * B[[i]] + sigma^2 * i * (T/steps) /2] * T/steps];
          AppendTo[Xanalytic, Exp[sigma * B[[i + 1]] - sigma^2 * (i + 1) * (T/steps) /2] * (X0 + IntB[[i + 1]])]];
         ListPlot[Xanalytic, PlotJoined → True, PlotRange → {-0.51, 20.1},
          Ticks → {{{1000, ""}, {2000, "2"}, {3000, ""}, {4000, "4"}, {5000, ""},
             {6000, "6"}, {7000, ""}, {8000, "8"}, {9000, ""}, {10000, "10"}}, Automatic}]
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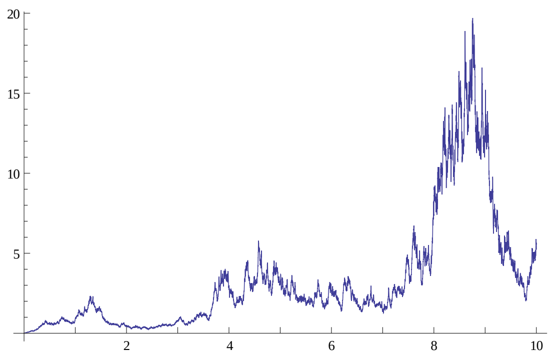
Out[36]=



Exercise 5. Exercise 4 in Stig Larsson's lecture notes.

Solution. In order to prove (37) we note that the first equation in the proof of Theorem 6 holds with T replaced by any $s \in [t,T]$. As the expectation of the martingale on the right-hand side of that equation is zero it follows that $\mathbf{E}\{u(Z(s;x,t),s)\} = \mathbf{E}\{u(Z(t;x,t),t)\}$ for $s \in [t,T]$. Recalling that $u(Z(t;x,t),t) = u(x,t)$ since $Z(t;x,t) = x$ is the initial value of the SDE that is posted just before the proof, we arrive at $\mathbf{E}\{u(Z(s;x,t),s)\} = u(x,t)$ for $s \in [t,T]$ [which is the version of (37) we prove].

Exercise 6. Exercise 5 in Larsson's lecture notes.

Solution. This follows in the following elementary fashion from integration by parts together with the fact that the functions ϕ and ψ involved have compact support in $\mathbb{R}\times(0,T)$ [that is, they are zero outside a closed and bounded set in $\mathbb{R}\times(0,T)$]:

$$
\begin{aligned}
&\int_0^T\int_{-\infty}^{\infty}(L\phi(x,t))\,\psi(x,t)\,dxdt\\
&=\int_0^T\int_{-\infty}^{\infty}\bigg(\frac{\partial\phi(x,t)}{\partial t}+\frac{\partial}{\partial x}\big(\mu(x,t)\phi(x,t)\big)-\frac{\partial^2}{\partial x^2}\frac{\sigma(x,t)^2\phi(x,t)}{2}\bigg)\psi(x,t)\,dxdt\\
&=\int_{-\infty}^{\infty}\Big[\phi(x,t)\psi(x,t)\Big]_{t=0}^{t=T}dx-\int_{-\infty}^{\infty}\int_0^T\phi(x,t)\,\frac{\partial\psi(x,t)}{\partial t}\,dtdx\\
&\quad+\int_0^T\Big[\mu(x,t)\phi(x,t)\psi(x,t)\Big]_{x=-\infty}^{x=\infty}dt-\int_0^T\int_{-\infty}^{\infty}\mu(x,t)\phi(x,t)\,\frac{\partial\psi(x,t)}{\partial x}\,dxdt\\
&\quad-\int_0^T\bigg[\bigg(\frac{\partial}{\partial x}\frac{\sigma(x,t)^2\phi(x,t)}{2}\bigg)\psi(x,t)\bigg]_{x=-\infty}^{x=\infty}dt+\int_0^T\int_{-\infty}^{\infty}\bigg(\frac{\partial}{\partial x}\frac{\sigma(x,t)^2\phi(x,t)}{2}\bigg)\frac{\partial\psi(x,t)}{\partial x}\,dxdt\\
&=0-\int_{-\infty}^{\infty}\int_0^T\phi(x,t)\,\frac{\partial\psi(x,t)}{\partial t}\,dtdx\\
&\quad+0-\int_0^T\int_{-\infty}^{\infty}\phi(x,t)\,\mu(x,t)\frac{\partial\psi(x,t)}{\partial x}\,dxdt\\
&\quad-0+\int_0^T\bigg[\frac{\sigma(x,t)^2\phi(x,t)}{2}\,\frac{\partial\psi(x,t)}{\partial x}\bigg]_{x=-\infty}^{x=\infty}dt-\int_0^T\int_{-\infty}^{\infty}\frac{\sigma(x,t)^2\phi(x,t)}{2}\,\frac{\partial^2\psi(x,t)}{\partial x^2}\,dxdt\\
&=\int_0^T\int_{-\infty}^{\infty}\phi(x,t)\bigg(-\frac{\partial\phi(x,t)}{\partial t}-\mu(x,t)\frac{\partial\psi(x,t)}{\partial x}-\frac{\sigma(x,t)^2}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}\bigg)\,dxdt\\
&=\int_0^T\int_{-\infty}^{\infty}\phi(x,t)\,(L^{\star}\psi(x,t))\,dxdt.
\end{aligned}
$$

Note that it is enough to require that one of the functions ϕ and ψ have compact support for the proof above to work: This will be important in the next exercise.

Exercise 7. Exercise 6 in Larsson's lecture notes.

Solution. Using (39) and (40) we shall prove that $u(x,t)$ given by (38) solves (34). [Note that Theorem 6 ensures that solutions to (34) are unique when they exist.] First note that (40) ensures that the terminal value $u(x,T)$ is $g(x)$ as required. In the language of Exercise 5 in Larsson's lecture notes we shall prove that $L^\star u(x,t) = 0$ for $u(x,t)$ given by (38). To that end we note that Exercise 5 in Larsson's lecture notes shows that for any function $\phi(x,t)$ that has compact support and satisfies $L\phi(x,t) = 0$ we have

$$\begin{aligned}\int_0^T \int_{-\infty}^{\infty} \phi(x,t)\,(L^\star u(x,t))\,dxdt &= \int_{-\infty}^{\infty}\int_0^T \int_{-\infty}^{\infty} \phi(x,t)\,(L^\star p(y,T,x,t)g(y))\,dxdtdy \\ &= \int_{-\infty}^{\infty}\int_0^T \int_{-\infty}^{\infty} (L\phi(x,t))\,p(y,T,x,t)g(y)\,dxdtdy \\ &= 0.\end{aligned}$$

Assuming that the space of functions $\phi(x,t)$ with compact support that satisfy $L\phi(x,t) = 0$ is rich enough, this proves that $L^\star u(x,t) = 0$.