TMS 165/MSA350 Stochastic Calculus

Home Exercises for Chapter 4 in Klebaner's Book

Througout this set of exercises $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. Show that a sequence $\{X_n\}_{n=1}^{\infty}$ of random variables such that $\mathbf{E}\{X_n^2\} < \infty$ for all n converges in $\mathbb{L}^2$ to some random variable X if and only if the limit $\lim_{m,n\to\infty} \mathbf{E}\{X_m X_n\}$ exists.

Task 2. Show the isometry property Equation 4.12 in Klebaner's book for the Itô integral process $\{\int_0^t X\, dB\}_{t\in[0,T]}$ for $X \in E_T$, e.g., using that the property holds for $X \in S_T$ [cf. Equation 4.5 in Klebaner's book].

Task 3. Show that for an $X \in P_T$ we have in the sense of convergence in probability

$$\int_0^T (X_n(t) - X(t))^2\, dt \to 0 \quad \text{as} \quad n \to \infty \quad \text{for some sequence} \quad \{X_n\}_{n=1}^{\infty} \subseteq S_T,$$

and that the Itô integral process $\{\int_0^t X\, dB\}_{t\in[0,T]}$ is well-defined as a limit in the sense of convergence in probability of $\int_0^t X_n\, dB$ as $n \to \infty$ for $t \in [0,T]$.

Task 4. Show that for a process $X \in P_T$ we have

$$\mathbf{P}\left\{\int_0^T X(t)^2\, dt = 0\right\} = 1 \;\Leftrightarrow\; \mathbf{P}\left\{\int_0^t X\, dB = 0\right\} = 1 \quad \text{for} \;\; t \in [0,T].$$

Task 5. Find stochastic processes $\{X(t)\}_{t\in[0,1]}$, $\{Y(t)\}_{t\in[0,1]}$ and $\{Z(t)\}_{t\in[0,1]}$ that belong to E_1, $P_1 \backslash E_1$ and P_1^c, respectively.

Task 6. Apply Itô's formula Theorem 4.17 in Klebaner's book to $f(X(t), Y(t))$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x,y) = g(x)\, y$ for a two times continuously differentiable function $g : \mathbb{R} \to \mathbb{R}$ and $X = Y = B$. Compare with what you get from applying the integration by parts formula Equation 4.57 in Klebaner's book with $X = g(B)$ and $Y = B$. Derive from the comaprison a new proof (without any explicit calculations other than applications of Itô's formula) of the property established in Example 4.23 in Klebaner's book that $[g(B), B](t) = \int_0^t g'(B(s))\, ds$ for $t \geq 0$.