TMS165/MSA350 Stochastic Calculus

Written Exam Friday 2 January 2015 $2-6$ pm

TEACHER AND JOUR: Patrik Albin, telephone 070 6945709. AIDS: None.

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t \geq 0}$ denotes a **P**-Brownian motion.

Task 1. Show that $B(t+T) - B(T)$, $c\,B(t/c^2)$ and $t\,B(1/t)$ are Brownian motions for constants $T, c > 0$. **(5 points)**

Task 2. A stochastic process $\{X(t)\}_{t \geq 0}$ with values in $(0,1)$ has stochastic differential with diffusion coefficient $\sigma(x) = x\,(1-x)$. Find the diffusion coefficient of the process $Y(t) = f(X(t)) = \ln(X(t)/(1-X(t))$. **(5 points)**

Task 3. Let $X(t)$ satisfy the SDE $dX(t) = \sqrt{X(t)+1}\, dB(t)$ for $t > 0$ with $X(0) = 0$. Assuming that Itô integrals are martingales, find $\mathbf{E}\{X(t)^2\}$. **(5 points)**

Task 4. Let a diffusion have $\sigma(x) = 1$ and $\mu(x) = -\frac{1}{2}\,\mathrm{sign}(x)$. Find the stationary probability density function $\pi(x)$. **(5 points)**

Task 5. Find $d\mathbf{Q}/d\mathbf{P}$ when $X(t) = B(t) + \sin(t)$ for $t \in [0, T]$ and $\mathbf{Q}$ is an equivalent probability measure to $\mathbf{P}$ such that $X(t)$ is a $\mathbf{Q}$-Brownian motion. **(5 points)**

Task 6. Derive the first order (/the first step) of the Itô-Taylor expansion with remainder for a time homogeneous SDE. **(5 points)**

TMS165/MSA350 Stochastic Calculus

Solutions to Written Exam 2 January 2015

Task 1. Note that the given three processes are all Gaussian with the same mean and covariance function 0 and $\min(s,t)$, respectively, as Brownian motion.

Task 2. The diffusion coefficient of $Y(t)$ is $f'(x) = \sigma(x)\,\frac{d}{dx}(\ln(x) - \ln(1-x)) = x\,(1-x)\,(1/x + 1/(1-x)) = 1$.

Task 3. This is Exercise 5.12 in Klebaner's book – according to his solution we have $\mathbf{E}\{X(t)^2\} = t$.

Task 4. By insertion in Equation 6.69 in Klebaner's book we find that $\pi(x) = \frac{1}{2}\,\mathrm{e}^{-|x|}$.

Task 5. This is Exercise 10.4 in Klebaner's book – according to his solution we have $d\mathbf{Q}/d\mathbf{P} = \exp\big\{-\int_0^T \cos(s)\,dB(s) - \frac{1}{2}\int_0^T \cos^2(s)\,ds\big\}$.

Task 6. See Section 2.4 in the lecture notes by Stig Larsson.